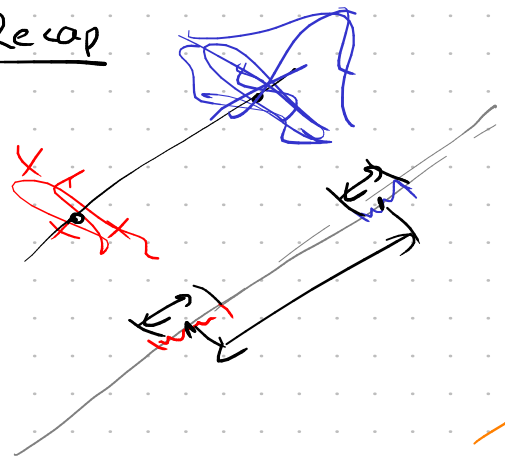
① Recap



generative model

$$p(C_1|\bar{x}) = \frac{p(\bar{x}|C_1)p(C_1)}{\sum_i \dots}$$

discriminative

② Logistic regression

C_1, C_2

$$p(C_1|\bar{x}) = \dots(\bar{w}^T\bar{x})\dots$$

$p(C_1|\bar{x}) \not\sim \bar{w}^T\bar{x}$

$\in [0,1]$

$$p(C_1|\bar{x}) = \frac{p(\bar{x}|C_1)p(C_1)}{p(\bar{x}|C_1)p(C_1)+p(\bar{x}|C_2)p(C_2)} =$$

$$= \frac{1}{1+\frac{p(\bar{x}|C_2)p(C_2)}{p(\bar{x}|C_1)p(C_1)}} = \frac{p(C_2|\bar{x})}{p(C_1|\bar{x})}$$

$\in (0,\infty)$

$\not\sim \bar{w}^T\bar{x}$

odds

$$p(C_1|\bar{x}) = \frac{1}{1+e^{-\log\frac{p(\bar{x}|C_1)p(C_1)}{p(\bar{x}|C_2)p(C_2)}}}$$

$\in (-\infty,\infty)$

log-odds

$\approx \bar{w}^T\bar{x}$

$$p(C_1|\bar{x}) = \frac{1}{1+e^{-\bar{w}^T\bar{x}}}; \quad p(C_2|\bar{x}) = 1 - \frac{1}{1+e^{-\bar{w}^T\bar{x}}}$$

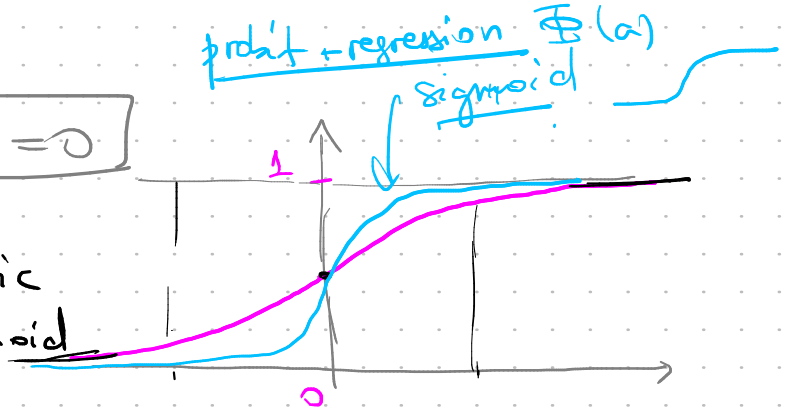
$$p(C_1|\bar{x}) = \frac{1}{2}, \quad \bar{w}^T\bar{x} = 0$$

probit regression $\Phi(a)$

sigmoid

$$\sigma(a) = \frac{1}{1+e^{-a}}$$

logistic sigmoid



$$1-\sigma(a) = \frac{e^{-a}}{1+e^{-a}} = \frac{1}{1+e^{a}} = \sigma(-a)$$

$$\sigma'(a) = \frac{+e^{-a}}{(1+e^{-a})^2} = \frac{e^{-a}}{1+e^{-a}} \cdot \frac{1}{(1+e^{-a})}$$

$$\boxed{\sigma'(a) = \sigma(a)\cdot(1-\sigma(a))}$$

$$e^{\log p(C_k)p(\bar{x}|C_k)} = p(\bar{x}|C_k)p(C_k)$$

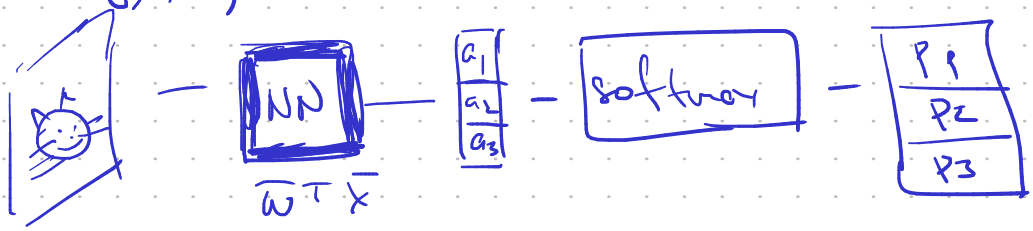
$C_1, \dots, C_K$ $\qquad p(C_k|\bar{x}) = \dfrac{p(\bar{x}|C_k)p(C_k)}{\sum_s p(\bar{x}|C_s)p(C_s)}$

$$\underbrace{\bar{w}_k^T \bar{x}}_{a_k} \approx \log p(C_k)p(\bar{x}|C_k)$$

affinity score

$$\text{softmax}(a_1, a_2, \dots, a_K) = \left(\dots, \frac{e^{a_k}}{\sum_s e^{a_s}}, \dots\right)$$

(cat, dog, fish)



$$t_n = \begin{cases} 1, & C_1 \\ 0, & C_2 \end{cases}$$

$$D = \{(\bar{x}_n, t_n)\}_{n=1}^{N}$$

(2) Maximum likelihood for LR

$\bar{w}^T\bar{x}$ $\qquad p(C_1|\bar{x}) = \sigma(\bar{w}^T\bar{x}), \quad p(C_2|\bar{x}) = 1-\sigma(\bar{w}^T\bar{x})$

$$p(D|\bar{w}) = \prod_n p(t_n|\bar{w}, \bar{x}_n) = \prod_n \begin{cases} \text{if } t_n=1 \Rightarrow \sigma(\bar{w}^T\bar{x}_n) \\ \text{if } t_n=0 \Rightarrow 1-\sigma(\bar{w}^T\bar{x}_n) \end{cases}$$

$$= \prod_n \left(t_n \sigma(\bar{w}^T\bar{x}_n) + (1-t_n)(1-\sigma(\bar{w}^T\bar{x}_n)) \right) =$$

$$= \prod_n \sigma(\bar{w}^T\bar{x}_n)^{t_n} \left(1-\sigma(\bar{w}^T\bar{x}_n)\right)^{1-t_n}$$

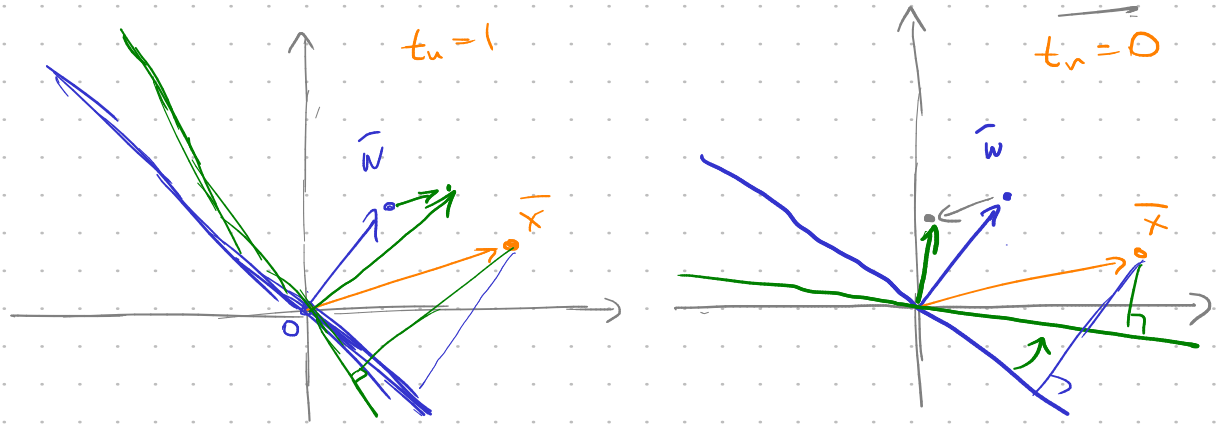
$$\log p(D|\bar{w}) = \sum_n \left(t_n \log \sigma_n + (1-t_n) \log(1-\sigma_n) \right)$$

$$\nabla_{\bar{w}} \log p(D|\bar{w}) = \sum_n \left(t_n \frac{\sigma_n(1-\sigma_n)}{\sigma_n} \bar{x}_n - (1-t_n) \frac{\sigma_n(1-\sigma_n)}{1-\sigma_n} \cdot \bar{x}_n \right)$$

$t_n - t_n\sigma_n - \sigma_n + t_n\sigma_n$

$$= \sum_n \left(t_n(1-\sigma_n) - (1-t_n)\sigma_n \right) \bar{x}_n = \sum_{n=1}^{N} \left(t_n - \sigma(\bar{w}^T \bar{x}_n) \right) \bar{x}_n$$

wpad. błąd — wpadnięcie

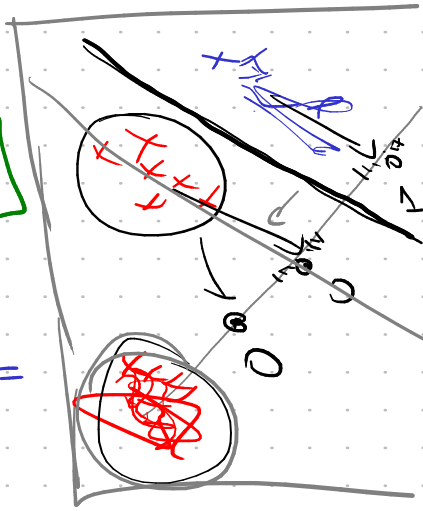


hessian

$$f(\bar{w}) \approx f(\bar{w}_0) + \nabla_{\bar{w}} f(\bar{w}_0)^T (\bar{w} - \bar{w}_0) + \frac{1}{2} (\bar{w} - \bar{w}_0)^T H \Big|_{\bar{w}_0} (\bar{w} - \bar{w}_0)$$

$$\nabla_{\bar{w}} f(\bar{w}_0) + H(\bar{w}_* - \bar{w}_0) = 0$$

$$\boxed{\bar{w}_* = \bar{w}_0 - H^{-1} \nabla_{\bar{w}} f(\bar{w}_0)}$$



$$\frac{\partial^2 \log p(D|\bar{w})}{\partial w_i \partial w_j} = \frac{\partial}{\partial w_i} \left(\sum_n \left(t_n - \sigma(\bar{w}^T \bar{x}_n) \right) x_{nj} \right) =$$

$$= \sum_n \left(-\sigma_n(1-\sigma_n) x_{ni} \right) x_{nj} = -\sum_n \sigma_n(1-\sigma_n) x_{ni} x_{nj}$$

$$H = -X^T \begin{pmatrix} \sigma_1(1-\sigma_1) & & 0 \\ & \sigma_2(1-\sigma_2) & \\ 0 & & \ddots \\ & & & \sigma_N(1-\sigma_N) \end{pmatrix} X = -X^T R X$$

IRLS

$$\boxed{\bar{w}^{(k+1)} = \bar{w}^{(k)} + (X^T R X)^{-1} X^T (\bar{t} - \sigma(X \bar{w}^{(k)}))}$$

$$p(\bar{w}|D) \propto p(\bar{w})\, p(D|\bar{w})$$

$$\log p(\bar{w}|D) = \text{const} + \log p(D|\bar{w}) + \log p(\bar{w})$$

$\propto \sum w_i^2$, $\propto \cdot \sum |w_i|$

(4) Multiclass LR

$C_1 \dots C_K$, $\bar{w}_1, \dots, \bar{w}_K$, $a_j = \bar{w}_j^T \bar{x}$

$$p(D|W) = \prod_{n=1}^{N} \prod_{k=1}^{K} p(C_k|\bar{x}_n, \bar{w})^{t_{nk}}$$

$$y_{nk} = \frac{e^{\bar{w}_k^T \bar{x}_n}}{\sum_j e^{\bar{w}_j^T \bar{x}_n}} \quad \left(\bar{w}_k^T \bar{x}_n = a_k,\ \bar{w}_j^T \bar{x}_n = a_j\right)$$

one-hot encoding: $\bar{t}_n = (0, \dots, 0, 1, 0, \dots, 0)$

$$\log p(D|W) = \sum_n \sum_k t_{nk} \log y_{nk}$$

$$\nabla_{\bar{w}_j} \log p(D|W) = \sum_n \sum_k t_{nk} \frac{1}{y_{nk}} \cdot y_{nk}\left([k=j] - y_{nj}\right)\bar{x}_n$$

$$\nabla_{\bar{w}_j} y_{nk} = \frac{\partial y_{nk}}{\partial a_j} \cdot \bar{x}_n$$

$$\frac{\partial y_{nk}}{\partial a_j} = \frac{\partial\left(e^{a_k} / \sum e^{a_s}\right)}{\partial a_j} =$$

$k=j$: $\frac{e^{a_k} \sum_s e^{a_s} - e^{a_k} \cdot e^{a_k}}{\left(\sum_s e^{a_s}\right)^2} =$

$= \frac{e^{a_k}}{\sum_s e^{a_s}} \cdot \left(1 - \frac{e^{a_k}}{\sum_s e^{a_s}}\right) =$

$= y_{nk}(1 - y_{nk})$

$= y_{nj}(1 - y_{nj})$

$k \neq j$: $-\frac{e^{a_k} \cdot e^{a_j}}{\left(\sum_s e^{a_s}\right)^2} =$

$= -y_{nk} y_{nj}$

$$= \begin{cases} -y_{nk} y_{nj}, & k \neq j \\ y_{nk}(1 - y_{nj}), & k = j \end{cases} =$$

$$= y_{nk}\left([k=j] - y_{nj}\right)$$

$$\nabla_{\bar{w}_j} \log p(D|\bar{w}) = \sum_n \left(t_{nj}(1 - y_{nj}) - \sum_{k \neq j} t_{nk} y_{nj}\right)\bar{x}_n =$$

$$= \sum_n \left(t_{nj} - \left(\sum_k t_{nk} y_{nj} \right) \right) \bar{x}_n = \sum_n \left(t_{nj} - y_{nj} \right) \bar{x}_n$$

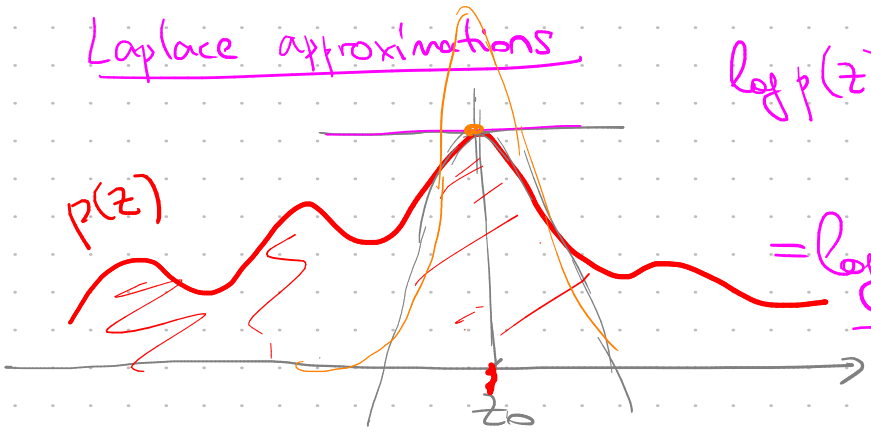
(1)

(5) Predictive distribution for LR

$$p(C_1 | \bar{x}, D) = \int p(C_1 | \bar{x}, \bar{w}) \, p(\bar{w} | D) \, d\bar{w}$$

$$= \int \sigma(\bar{w}^T \bar{x}) \, p(\bar{w} | D) \, d\bar{w} = \mathbb{E}_{p(\bar{w}|D)} [\sigma(\bar{w}^T \bar{x})]$$

Laplace approximation



$$\log p(z) \approx \log p(z_0) + \frac{1}{2} \frac{\partial^2 \log p}{\partial z^2} \Big|_{z_0} (z - z_0)^2$$

$$= \log p(z_0) - \frac{1}{2} A \cdot (z - z_0)^2$$

$$A = - \frac{\partial^2}{\partial z^2}$$

$$p(z) \approx p(z_0) \cdot e^{-\frac{1}{2} A (z - z_0)^2} \approx N\left(z \,|\, z_0, \frac{1}{A}\right)$$

$$\log p(\bar{z}) \approx \log p(\bar{z}_0) - \frac{1}{2} (\bar{z} - \bar{z}_0)^T A (\bar{z} - \bar{z}_0)$$

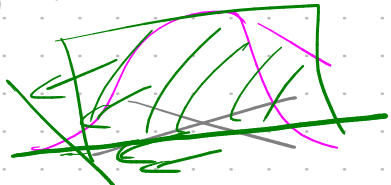
$$\boxed{A = - \nabla \nabla \log p}$$

$$p(C_1 | D, \bar{x}) = \int \sigma(\bar{w}^T \bar{x}) \cdot p(\bar{w} | D) \, d\bar{w} \approx \left[\bar{w}_{MAP} \right] \approx$$

$$\approx \int \sigma(\bar{w}^T \bar{x}) \cdot N(\bar{w} \,|\, \bar{w}_{MAP}, \Sigma_N) \, d\bar{w}$$

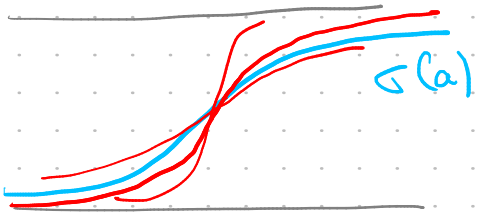
$\bar{w}^T \bar{x} = a$

$$\sigma(\bar{w}^T \bar{x}) = \int_{-\infty}^{\infty} \sigma(a) \cdot \delta(\bar{w}^T \bar{x} - a) \, da$$



$$= \int_{-\infty}^{\infty} \sigma(a) \left(\int \delta(\bar{w}^T \bar{x} - a) \mathcal{N}(\bar{w} \mid \bar{w}_{MAP}, \Sigma_N) \, d\bar{w} \right) da$$

$$= \int_{-\infty}^{\infty} \sigma(a) \, \mathcal{N}(a \mid \mu_a, \sigma_a^2) \, da \approx$$



probit function $\Phi(b) = \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$\sigma(a) \approx \Phi\left(\sqrt{\frac{\pi}{8}}\, a\right)$ $\quad \Phi(\lambda a) \approx \sigma(a)$

λa

$$\approx \int_{-\infty}^{\infty} \int_{-\infty} \mathcal{N}(x \mid 0, 1) \cdot \mathcal{N}(a \mid \mu_a, \sigma_a^2) \, dx \, da =$$

$$= \Phi\left(\frac{\mu_a}{\sqrt{\sigma_a^2 + 1/\lambda^2}} \right) = \Phi\left(\frac{\lambda \mu_a}{\sqrt{1 + \lambda^2 \sigma_a^2}} \right) \approx$$

$= \bar{w}_{MAP}^T \bar{x}$

$$\approx \sigma\left(\frac{\mu_a}{\sqrt{1 + \frac{\pi}{8} \sigma_a^2}} \right)$$

