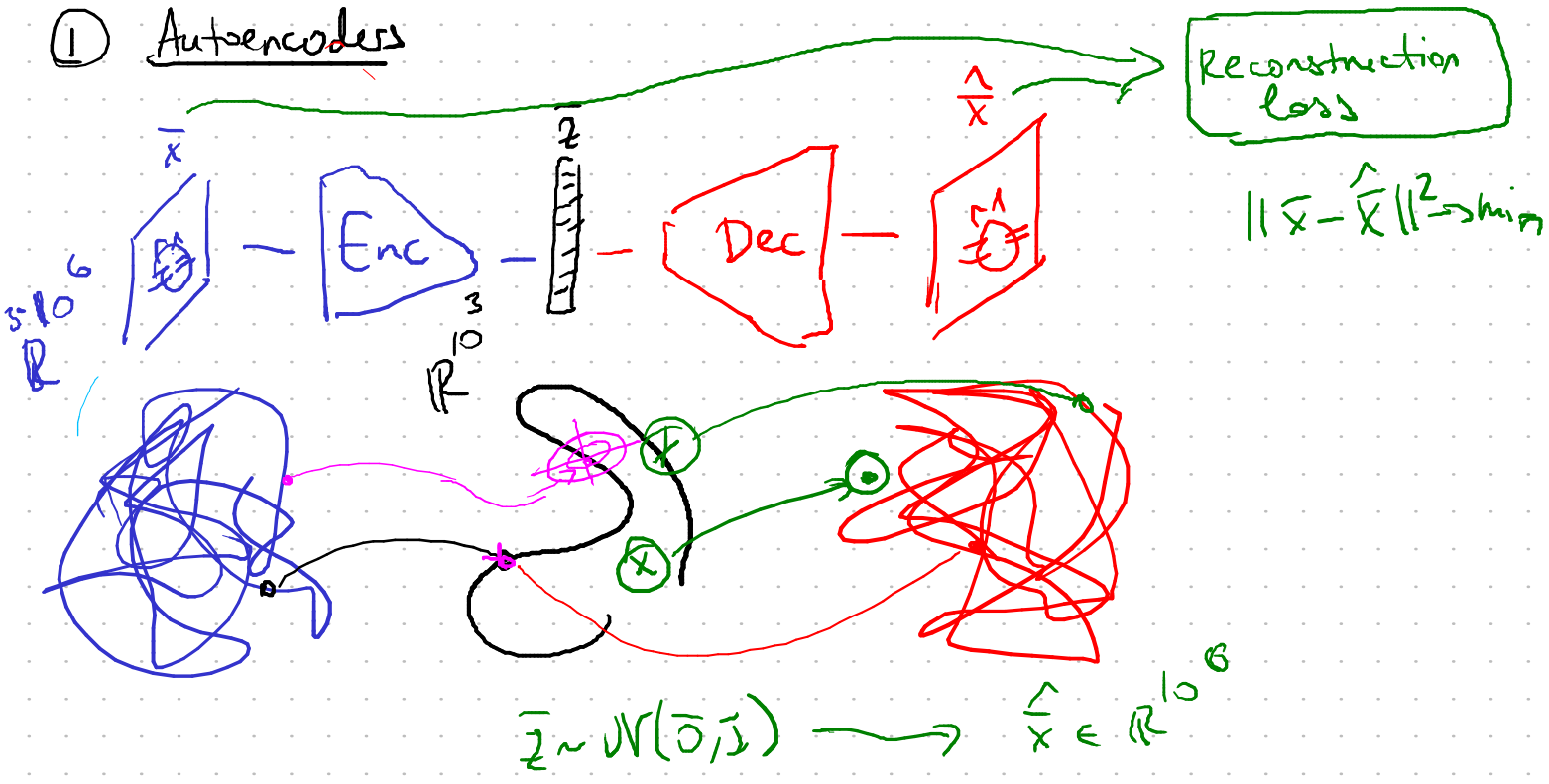
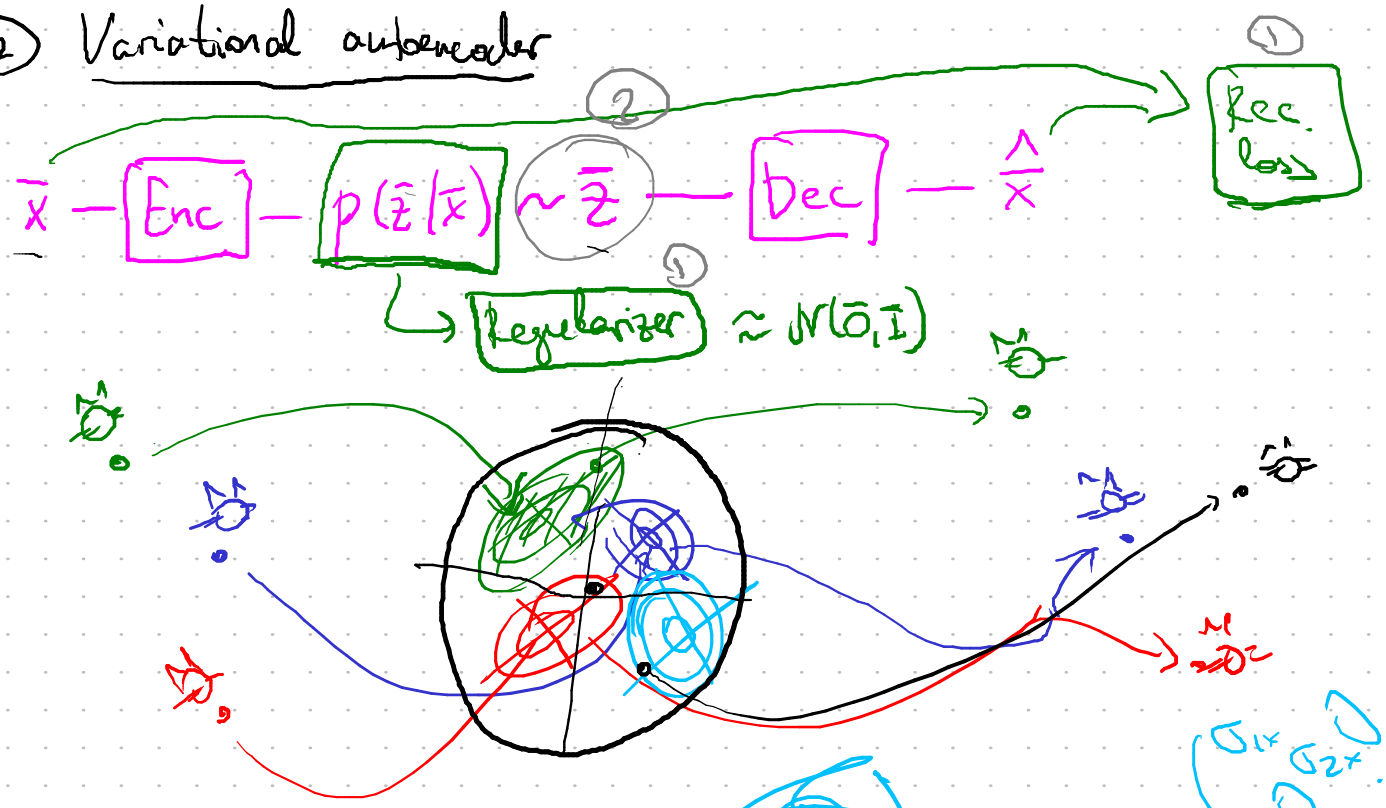
① Autoencoders



② Variational autoencoder



③ VAE, probabilistic view



$$\underbrace{p(\bar{z})}_{\text{prior}}\underbrace{p(\bar{x}|\bar{z})}_{\text{decoder}} = \underbrace{p(\bar{x},\bar{z})}_{\text{generation}} = p(\bar{x})\underbrace{p(\bar{z}|\bar{x})}_{\text{encoder}}$$

$$p(\bar{z})=\mathcal{N}(\bar{0},\bar{I})$$

$$p(\bar{x}|\bar{z})=\mathcal{N}\left(\bar{x}\,|\,Dec(\bar{z}),\,c\cdot\bar{I}\right)$$

$$p(\bar{z}|\bar{x})\approx q(\bar{z})=\mathcal{N}\left(\bar{z}\,|\,\bar{\mu}_x,\bar{\sigma}_x\right)$$

$$\begin{pmatrix}\sigma_{1x} & & \\ & \sigma_{2x} & \\ & & \ddots \\ & & & \sigma_{dx}\end{pmatrix}$$

$$\log p(\bar{x}) = \log p(\bar{x},\bar{z}) - \log p(\bar{z}|\bar{x})$$

$\int \mathbb{E}_{q(\bar{z})}[\cdot]$

$$\log p(\bar{x}) = \mathbb{E}_{q(\bar{z})}\left[\log p(\bar{x},\bar{z}) - \log p(\bar{z}|\bar{x})\right] \pm \mathbb{E}_{q(\bar{z})}\left[\log q(\bar{z})\right]$$

$\int \log \frac{q}{p} \cdot q\,dx$

$\parallel$

$\mathbb{E}_q\left[\log \frac{q}{p}\right]$

Kullback-Leibler divergence

$$\log p(\bar{x}) = \underbrace{\mathbb{E}_{q(\bar{z})}\left[\log \frac{p(\bar{x},\bar{z})}{q(\bar{z})}\right]} + \underbrace{\mathbb{E}_{q(\bar{z})}\left[\log \frac{q(\bar{z})}{p(\bar{z}|\bar{x})}\right]}$$

$$\text{Const} = \underbrace{L(q)}_{\text{Variational lower bound ELBO}} + \underbrace{KL\left(q(\bar{z}) \,\|\, p(\bar{z}|\bar{x})\right)}$$

$q(\bar{z}) \Leftrightarrow$ max

$q(\bar{z}) \to$ min

$$L(q) = \int q(\bar{z}) \log \frac{p(\bar{x},\bar{z})}{q(\bar{z})}\, d\bar{z} \xrightarrow{q} \max$$

$= p(\bar{z})\, p(\bar{x}|\bar{z})$

$$= \int q(\bar{z}) \log \frac{p(\bar{z})\, p(\bar{x}|\bar{z})}{q(\bar{z})}\, d\bar{z} =$$

$$= \int q(\bar{z}) \log p(\bar{x}|\bar{z})\, d\bar{z} - \int q(\bar{z}) \log \frac{q(\bar{z})}{p(\bar{z})}\, d\bar{z} =$$

$$= \underbrace{\mathbb{E}_{q(\bar{z})}\left[\overbrace{\log p(\bar{x}|\bar{z})}^{\text{decoder}}\right]} - \underbrace{KL\left(q(\bar{z}) \,\|\, \overbrace{p(\bar{z})}^{\mathcal{N}(\bar{0},I)}\right)}_{L_{reg}}$$

$\parallel$

$\mathbb{E}_{q(\bar{z})}\left[\log \mathcal{N}(\bar{x} \mid \mathrm{Dec}(\bar{z}), cI)\right] =$

$$= \mathbb{E}_q \left[\text{Const} - \frac{1}{2c^2} \| \bar{x} - \text{Dec}(\bar{z}) \|^2 \right]$$

$$\simeq \text{Const} - \text{Const} \cdot \underbrace{\sum_n \| \bar{x}_n - \text{Dec}(\text{Enc}(\bar{x}_n)) \|^2}_{L_{rec}}$$