① Learning the uniform distribution

$1, \dots, N$ $\quad D = \{m_1, m_2, \dots, m_k\}$

$$p(m_i | N) = \begin{cases} 1/N, & m_i \le N \\ 0, & m_i > N \end{cases}$$

$N_{ML} = ?$

$$p(D|N) = \left(\frac{1}{N}\right)^k, \quad \forall i \; m_i \le N \xrightarrow{N} \max$$

$N_{ML} = \max_i m_i$

$M = \max_{i=1}^{k} m_i$ $\quad \mathbb{E}[M|N]$

$k=1$: $\mathbb{E}[m|N] = \frac{N+1}{2} = m$

$$\hat{N} = 2m - 1$$

$k > 1$.

$$p(M=m|N) = \frac{\#\{\text{выборок } k \text{ чисел } \max = m\}}{\#\{\text{выборок } k \text{ чисел от } 1 \text{ до } N\}}$$

$$p(M=m|N) = \frac{\binom{m-1}{k-1}}{\binom{N}{k}}$$

$\frac{m!}{(k-1)!(m-k)!} = \frac{(m-1)!}{(k-1)!(m-k)!} \cdot m$

$$\mathbb{E}[M|N] = \sum_{m=k}^{N} m \cdot p(M=m|N) = \frac{\sum_{m=k}^{N} m \binom{m-1}{k-1}}{\binom{N}{k}} =$$

$$= \frac{1}{\binom{N}{k}} \sum_{m=k}^{N} \frac{m!}{(k-1)!(m-k)!} = \frac{k}{\binom{N}{k}} \frac{\sum_{m=k}^{N} \binom{m}{k}}{\binom{N+1}{k+1}} =$$

$$\sum_{a=c}^{b+1} \binom{a}{c} = \sum_{a=c}^{b} \binom{a}{c} + \binom{b+1}{c} = \binom{b+1}{c+1} + \binom{b+1}{c} =$$

$$= \binom{b+2}{c+1}$$

$$= k \cdot \frac{(N+1)!}{(k+1)!\,(N-k)!} \cdot \frac{k!\,(N-k)!}{N!} = \frac{k(N+1)}{k+1} = \mathbb{E}[M_k]$$

$$\boxed{\hat{N} = \frac{k+1}{k} \cdot m - 1}$$

(2) Bayesian tanks

$$p(N \mid M, k) = \frac{p(M \mid N, k) \cdot p(N \mid k)}{p(M \mid k)} \quad \text{— ?}$$

$$p(M \mid k) = \sum_N p(M \mid N, k)\, p(N \mid k)$$

$$p(N \mid k) = \text{Unif}(k, \dots, \Omega) = \frac{1}{\Omega - k + 1}, \quad k \le N < \Omega$$

$$p(N \mid M, k) = \frac{\frac{1}{\Omega-k+1} \cdot \binom{M-1}{k-1} \Big/ \binom{N}{k}}{\sum_{n=M}^{\Omega} \frac{1}{\Omega-k+1} \binom{M-1}{k-1} \Big/ \binom{n}{k}}$$

$$p(N \mid M, k) = \frac{\binom{N}{k}^{-1}}{\sum_{n=1}^{\Omega} \binom{n}{k}^{-1}} = \frac{\binom{N}{k}^{-1}}{\binom{M}{k}^{-1} \cdot \frac{M}{k-1}}$$

improper prior

некорректное априорное распределение

$\Omega \to \infty$

— factorial distribution

Монетка: Unif $\to$ $p(\theta|D) \propto \theta^n (1-\theta)^m$

$B(1,1)$

$$p(\theta) = B(0,0) \propto \frac{1}{\theta(1-\theta)}$$ — improper prior

$$p(\theta|D) \propto \theta^{n-1}(1-\theta)^{m-1}$$

3 Counting the ducks

Mark and recapture: $\boxed{p\left(\begin{matrix}\text{поймать}\\ \text{утку}\end{matrix}\right) = \frac{n}{N}}$ — все

окольцованные

$$\frac{\overset{n}{\text{окольц}}}{\underset{N}{\text{все}}} = \frac{\overset{m}{\text{окольц. поймали}}}{\underset{M}{\text{поймали}}}$$

Lincoln – Petersen

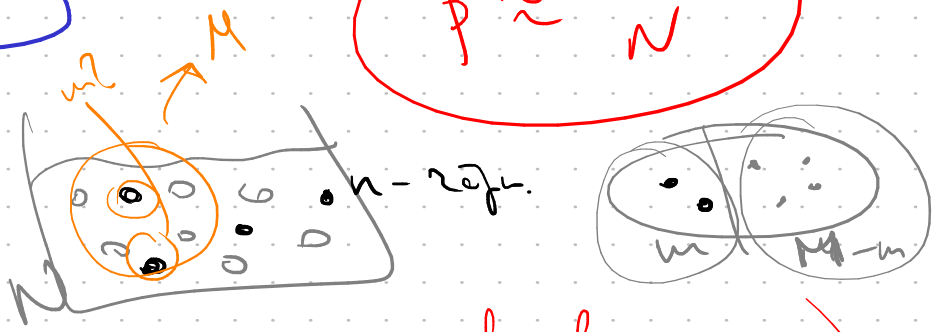
$$\boxed{N \approx \frac{n \cdot M}{m}}$$

$$p \approx \frac{n}{N}$$

Попул. N

всего окольц. n

выборка M



$$p(m|N,M,n) = p\left(\begin{matrix}\text{„}m\text{ успехов в выб. из }M\text{ эл-тов из}\\ \text{попул., где }n\text{ из }N\text{ успехов"}\end{matrix}\right)$$

$$= \frac{\binom{n}{m}\cdot\binom{N-n}{M-m}}{\binom{N}{M}}$$

hypergeometric distribution

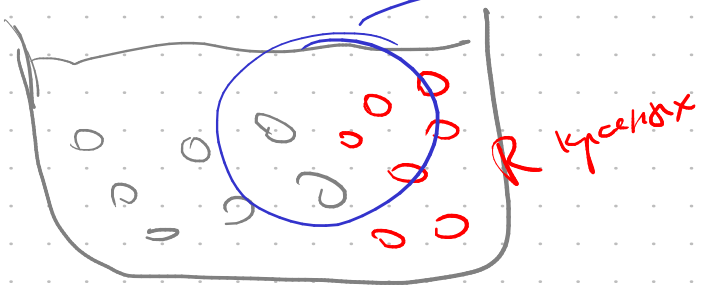
improper prior

$$p(N|m,M,n) = \frac{p(m|N,M,n)\cdot \cancel{p(N|M,n)}}{\sum_{N'} p(m|N',M,n)\cdot \cancel{p(N'|M,n)}}$$

$$p(\bar{\theta}|D) = \frac{p(D|\bar{\theta})\,p(\bar{\theta})}{p(D)}$$

$$p(\bar{\theta}|D) \propto p(D|\bar{\theta})$$

(4) Jaynes' example



$$p(z|N,R,n) = \frac{\binom{R}{z}\binom{N-R}{n-z}}{\binom{N}{n}}$$

$D = \{(n,z)\}$

N, R ?

$$p(N,R|D) = \frac{p(D|N,R)\,p(N,R)}{p(D)}$$

$$p(N,R) = p(N)\,p(R|N)$$

Предел:

$$p(N|D) = \begin{cases} \text{const} \cdot p(N), & N \geq n \\ 0, & N < n \end{cases}$$

$$p(N|D) = \frac{p(N)p(D|N)}{p(D)} = \frac{p(N)}{p(D)} \cdot \sum_{R=0}^{N} p(D|N,R)\,p(R|N)$$

$$p(D|N) = \sum_R p(D|N,R)\,p(R|N) = \begin{cases} f(n,z), & N \geq n \\ 0, & N < n \end{cases}$$

$$p(z|N,R,n) = \frac{\binom{R}{z}\binom{N-R}{n-z}}{\binom{N}{n}}$$

$$\sum_R \binom{R}{z}\binom{N-R}{n-z} \cdot p(R|N) = \binom{N}{n} \cdot f(n,z)$$