① Энтропия $(p_1 \dots p_n)$ „мера неопред."

$$H_n(p_1,\dots,p_n) = -\sum_i p_i \log p_i$$

$$h(mn) = h(m) + h(n)$$

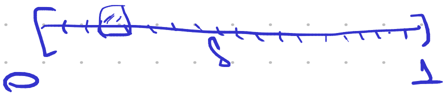
$$c \cdot \log n$$

- H_n – непр.
- $h(n) = H_n\left(\frac{1}{n} \dots \frac{1}{n}\right)$ – возр. по n
- непротив.

$$H_3(p_1, p_2, p_3) = H_2(p_1, p_2+p_3) + (1-p_1) \cdot H_2\left(\frac{p_2}{p_2+p_3}, \frac{p_3}{p_2+p_3}\right)$$

Про $(p_1, p_2, \dots, p_n)$ есть информ. I

N



$$\delta = \frac{1}{N}$$

N раз бросать честную игр. кость с n гранями

$$p_i = \frac{N_i}{N} \qquad N \text{ брос.} - N_1, N_2, \dots, N_n$$

$$\Pr(p_1 \dots p_n) = \frac{1}{n^N} \cdot \frac{N!}{N_1!\, N_2! \dots N_n!} \qquad \binom{N}{N_1\ N_2\ \dots\ N_n}$$

$$\xrightarrow[p_1 \dots p_n]{} \max$$

при усл. I

$$\log N! = N \log N - N + O(\sqrt{N})$$

$$\log \frac{N!}{N_1! \dots N_n!} = \underbrace{N \log N}_{= \sum N_i \log N} - \cancel{N} - \sum_{i=1}^{n} \left(N_i \log N_i - \cancel{N_i}\right) + O(\sqrt{N})$$

$$= -\sum_{i=1}^{n} \underbrace{N_i}_{= p_i \cdot N} (\log N_i - \log N) + O(\sqrt{N}) =$$

$$= N \cdot \underbrace{\left(-\sum_{i=1}^{n} p_i \log p_i\right)}_{\xrightarrow[p_1 - p_n]{} \max} + O(\sqrt{N})$$

② Maximum Entropy Principle

* кубик $\mathbb{E}[X] = m$

$$\underbrace{H = -\sum_{i=1}^{6} p_i \log p_i}\ \to \max$$

$$\underline{p_1 + \dots + p_6 = 1}, \quad p_i \geq 0$$

$$\underline{p_1 + 2p_2 + \dots + 6p_6 = m}$$

$$L = -\sum_{i=1}^{6} p_i \log p_i - \lambda\left(\sum i \cdot p_i - m\right) - \mu\left(\underline{\sum p_i - 1}\right) \to \max$$

$$\frac{\partial L}{\partial p_i} = -\log p_i - 1 - \lambda \cdot i - \mu = 0$$

$$\log p_i = -(\lambda i + \mu + 1), \quad \underline{p_i = e^{-(\lambda i + \mu + 1)}}$$

$$\sum p_i = 1 \qquad \sum e^{-\lambda i - \mu - 1} = \underline{e^{-\mu - 1} \cdot \underbrace{\sum e^{-\lambda i}}_{i}} = 1$$

$$\sum i p_i = m \qquad e^{-\mu-1} \sum_i i \cdot e^{-\lambda i} = m$$

$$\underline{m = \frac{\sum_i i \cdot e^{-\lambda i}}{\sum_i e^{-\lambda i}}}, \qquad \mu = \log \sum e^{-\lambda i} - 1$$

Найти $p(x)$, $x \in \mathbb{R}$ — max. ent. с зад. μ, σ^2

$$H(p) = -\int_{-\infty}^{\infty} p(x) \log p(x)\, dx \xrightarrow[p]{} \max$$

$$\int \underline{p(x) dx = 1} \qquad \int \underline{x\, p(x)\, dx} = \mu \qquad \int \underline{(x-\mu)^2} p(x) dx = \sigma^2$$

$$L(x, \lambda_0, \lambda_1, \lambda_2) = -\int_{-\infty}^{\infty} p(x) \underline{\log p(x) dx} - \underline{\lambda_0}\left(1 - \int \underline{p(x)} dx\right)$$

$$-\lambda_1\left(\underline{\mu} - \int x\, p(x) dx\right) - \lambda_2\left(\underline{\sigma^2} - \int (x-\mu)^2 p(x)\, dx\right)$$

$$p(x) + \delta p(x)$$

$$\delta L = \int_{-\infty}^{\infty} \delta p(x) \left(\log p(x) + 1 + \lambda_0 + \lambda_1 x + \lambda_2 (x-\mu)^2 \right) dx$$

$= 0$

$$0 = \log p(x) + (\_\, x^2 \,\_\, x \,\_\,\_)$$

$$p(x) = e^{-1-\lambda_0-\lambda_1 x - \lambda_2 (x-\mu)^2}$$

rayrcwalt

$$\boxed{p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}$$

$p(x) = \mathcal{N}$

$\forall q(x)$ $\quad H(p) \geq H(q)$ $\quad E[q] = \mu$ $\quad Var[q] = \sigma^2$

$$0 \leq KL(q \| p) = \int q \log \frac{q}{p} = -H(q) - \int_{-\infty}^{\infty} q(x) \log p(x)\, dx$$

$$\int q \log p \, dx = -\int q \cdot \log \sqrt{2\pi\sigma^2}\, dx - \frac{1}{2\sigma^2} \int q(x) (x-\mu)^2 dx =$$

$$= -\frac{1}{2} \log(2\pi\sigma^2) - \frac{\sigma_q^2}{2\sigma^2} = -\frac{1}{2} \log(2\pi\sigma^2) - \frac{1}{2}$$

$$H(q) \leq \frac{1}{2} \log(2\pi\sigma^2) + \frac{1}{2} = H(p)$$

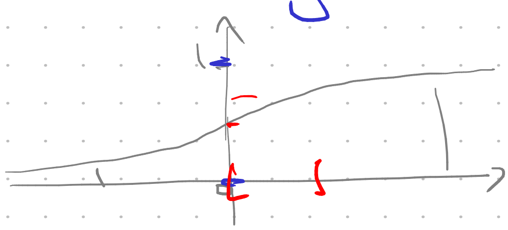
$$H(\mathcal{N}(x|\mu,\sigma^2)) = -E_{\mathcal{N}} \left(-\frac{1}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} (x-\mu)^2 \right) =$$

$$= \frac{1}{2} \log(2\pi\sigma^2) + \frac{1}{2\sigma^2} \cdot \sigma^2$$

③ Jeffreys' priors

uninformative prior $\quad \frac{1}{\sqrt{\theta(1-\theta)}}$

4 motivating $\quad \theta = p(\text{heads})$ $\quad \boxed{p_\theta(\theta) = Unif([0,1])}$



log-odds

η: $\quad \theta = \frac{1}{1+e^{-\eta}}, \quad e^{-\eta} = \frac{1-\theta}{\theta}, \quad \boxed{\eta = \log \frac{\theta}{1-\theta}}$

$p_\theta(\theta) = \begin{cases} 1, & [0,1] \\ 0 & -''- \end{cases}$ $\theta = \dfrac{1}{1+e^{-\eta}}$ improper prior

$$p_\eta(\eta) = p_\theta(\theta) \cdot |\theta'(\eta)| = \frac{e^{-\eta}}{(1+e^{-\eta})^2}$$

$p_\theta(\theta)$ для модели, $\theta \in \mathbb{R}$

$\eta = h(\theta)$

1) $p_\eta(\eta)$

2) $\eta = h(\theta) \Rightarrow \tilde{p}_\eta(\eta) = \dfrac{p_\theta(h^{-1}(\eta))}{|h'(h^{-1}(\eta))|}$ $\Big\}$ $p_\eta(\eta) = \tilde{p}_\eta(\eta)$

$$\boxed{\tilde{p}_\eta(h(\theta)) = \frac{p_\theta(\theta)}{|h'(\theta)|}}$$

$\int \sqrt{I(\theta)}\, d\theta < \infty$ если нет, то improper prior

модель $p_\theta(\bar{x}|\theta)$

$$\boxed{p^J(\theta) = \mathrm{Const} \cdot \sqrt{I(\theta)}}$$

$$I_\theta(\theta) = \int \left(\frac{\partial}{\partial\theta} \log p_\theta(\bar{x}|\theta)\right)^2 p_\theta(\bar{x}|\theta)\, dx =$$

$$= \int \left(\frac{\partial}{\partial\theta} \log p_\eta(\bar{x}|h(\theta))\right)^2 p_\eta(\bar{x}|h(\theta))\, d\bar{x} =$$

$$= \int \left(h'(\theta) \frac{\partial}{\partial\eta} \log p_\eta(\bar{x}|\eta)\right)^2 p_\eta(\bar{x}|\eta)\, d\bar{x} = (h'(\theta))^2 \cdot I_\eta(h(\theta))$$

$$p^J(\eta) = p^J(h(\theta)) = \mathrm{Const} \cdot \sqrt{I_\eta(\eta)} = \mathrm{Const} \cdot \frac{\sqrt{I_\theta(\theta)}}{|h'(\theta)|} = \frac{p^J(\theta)}{|h'(\theta)|}$$

4 Монетка и гауссиан

$$p(x|\theta) = \theta^{[x=1]} (1-\theta)^{[x=0]}$$

$[x=1] \cdot \log\theta + [x=0] \log(1-\theta)$

$$p^J(\theta) \propto \sqrt{I(\theta)} = \sqrt{\mathbb{E}_{p_\theta(x|\theta)}\left[\left(\frac{\partial}{\partial\theta}\log p(x|\theta)\right)^2\right]} =$$

$$= \sqrt{\mathbb{E}\left[\left(\frac{[x=1]}{\theta} - \frac{[x=0]}{1-\theta}\right)^2\right]} =$$

$$= \sqrt{\theta\cdot\left(\frac{1}{\theta}\right)^2 + (1-\theta)\cdot\left(\frac{-1}{1-\theta}\right)^2} = \sqrt{\frac{1}{\theta}+\frac{1}{1-\theta}} = \frac{1}{\sqrt{\theta(1-\theta)}}$$

$$\boxed{p^J(\theta) = \mathrm{Beta}\left(\theta \,\middle|\, \tfrac{1}{2}, \tfrac{1}{2}\right)}$$

$\ast\ \mathcal{N}(x|\mu,\sigma^2)$, $\sigma^2 = \text{Const}$

$$\cancel{-\frac{1}{\sqrt{2\pi\sigma^2}}} - \frac{1}{2\sigma^2}(x-\mu)^2 \qquad -2(x-\mu)$$

$$I(\mu) = \mathbb{E}_{\mathcal{N}(x|\mu)}\left[\left(\frac{\partial}{\partial\mu}\log p(x|\mu)\right)^2\right] =$$

$$= \mathbb{E}_{\mathcal{N}(x|\mu)}\left[\left(\frac{x-\mu}{\sigma^2}\right)^2\right] = \frac{1}{\sigma^4}\,\mathbb{E}_{(x|\mu)}\left[(x-\mu)^2\right] = \frac{\sigma^2}{\sigma^4} = \frac{1}{\sigma^2}$$

$$p^J(\mu) \propto \sqrt{I(\mu)} = \frac{1}{\sigma} = \text{Const}$$

(5) $\bar\theta, \bar\eta \in \mathbb{R}^d$

$$\boxed{p^J(\bar\theta) = \text{Const}\cdot\sqrt{\det I(\bar\theta)}}$$

$$p(\bar\eta) = p(\bar\theta)\cdot\left|\det\frac{\partial\theta_i}{\partial\eta_j}\right|$$

$$p^J(\bar\eta) = p^J(\bar\theta)\cdot\left|\det\frac{\partial\theta_i}{\partial\eta_j}\right| \propto \sqrt{\det I(\bar\theta)\left(\det\frac{\partial\theta_i}{\partial\eta_j}\right)^2} =$$

$$= \sqrt{\det\frac{\partial\theta_i}{\partial\eta_j}\cdot\det\mathbb{E}\left[\frac{\partial\log p_\theta(\bar x|\bar\theta)}{\partial\theta_k}\cdot\frac{\partial\log p_\theta(\bar x|\bar\theta)}{\partial\theta_l}\right]\cdot\det\frac{\partial\theta_i}{\partial\eta_j}} =$$

$$= \sqrt{\det\mathbb{E}\left[\sum_{k,l}\frac{\partial\theta_k}{\partial\eta_i}\frac{\partial\log p_\theta}{\partial\theta_k}\frac{\partial\log p_\theta}{\partial\theta_l}\cdot\frac{\partial\theta_l}{\partial\eta_j}\right]}$$

$$\sum_k \frac{\partial\theta_k}{\partial\eta_i}\frac{\partial\log}{\partial\theta_k} = \frac{\partial\log p}{\partial\eta_i}$$

$$= \sqrt{\det E\left[\frac{\partial \log p_\eta(\bar{x}|\bar{\eta})}{\partial \eta_i} \cdot \frac{\partial \log p_\eta(\bar{x}|\bar{\eta})}{\partial \eta_j} \right]} = \sqrt{\det I(\bar{\eta})}$$

$$p_{\bar{\theta}}(x) = \mathcal{N}(x \mid \mu, \sigma^2) \qquad \bar{\theta} = \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}$$

$$\boxed{I(\theta) = \left(\frac{\partial \log p}{\partial \theta_i} \cdot \frac{\partial \cdots}{\partial \theta_j} \right) = \boxed{- \left(\frac{\partial^2 \log p}{\partial \theta_i \partial \theta_j} \right)}}$$

$$I(\mu, \sigma^2) = -E\left[\begin{pmatrix} \frac{\partial^2 \log p}{\partial \mu^2} & \frac{\partial^2 \log p}{\partial \mu \partial \sigma^2} \\ \frac{\partial^2 \log p}{\partial \mu \partial \sigma^2} & \frac{\partial^2 \log p}{(\partial \sigma^2)^2} \end{pmatrix} \right]$$

$$\log p(x \mid \mu, \sigma^2) = -\frac{1}{2} \log \sigma^2 - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$\frac{\partial \log p}{\partial \mu} = \frac{(x-\mu)}{\sigma^2} \qquad \frac{\partial \log p}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{(x-\mu)^2}{2(\sigma^2)^2}$$

$$\frac{\partial^2 \log p}{\partial \mu^2} = -\frac{1}{\sigma^2}, \qquad \frac{\partial^2 \log p}{\partial \mu \partial \sigma^2} = -\frac{(x-\mu)}{\sigma^4}$$

$$\frac{\partial^2 \log p}{(\partial \sigma^2)^2} = \frac{1}{2\sigma^4} + \frac{(x-\mu)^2}{\sigma^6}$$

$$I(\bar{\theta}) = -E_x\left[\begin{pmatrix} -1/\sigma^2 & -(x-\mu)/\sigma^4 \\ - \cdots - & \frac{1}{2\sigma^4} + \frac{(x-\mu)^2}{\sigma^6} \end{pmatrix} \right] =$$

$$= - \begin{pmatrix} -\frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} - \frac{1}{\sigma^4} \end{pmatrix} = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/2\sigma^4 \end{pmatrix}$$

$$\det I(\bar{\theta}) = \frac{1}{2\sigma^6}$$

$$\boxed{p(\underset{\in \mathbb{R}}{\mu}, \underset{\in \mathbb{R}+}{\sigma^2}) \propto \frac{1}{\sigma^3}}$$

inverse chi-squared distribution