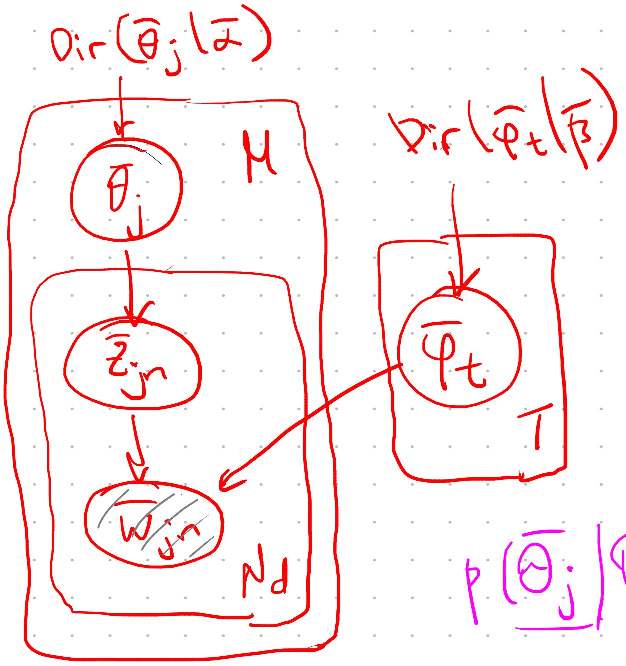
# ① Gibbs sampling for LDA




$$p(\Theta, \Phi, Z \mid W, \alpha, \beta) \qquad \text{Var.appr.} \quad \approx q(\Theta, \Phi, Z)$$

$$p(\Theta, \Phi, Z \mid W, \alpha, \beta)$$

1) $\bar\theta_j \sim p(\bar\theta_j \mid \Theta_{-j}, \Phi, Z, W, \bar\alpha, \bar\beta)$

$$p(\bar\theta_j \mid \Theta_{-j}, \Phi, Z, W, \bar\alpha, \bar\beta) \propto$$

$$\propto p(\Theta, \Phi, Z, W \mid \bar\alpha, \bar\beta) = \prod Dir(\bar\theta_i \mid \bar\alpha) \prod_t Dir(\bar\varphi_t \mid \bar\beta) \times \prod_{i=1}^{M} \prod_{n=1}^{N_i} Mult(\bar z_{in} \mid \bar\theta_i) \cdot Mult(\bar w_{in} \mid \varphi_{\bar z_{in}})$$

$$\propto Dir(\bar\theta_j \mid \bar\alpha) \cdot \prod_{n=1}^{N_j} Mult(\bar z_{jn} \mid \bar\theta_j) \propto$$

$$\propto \prod_t \theta_{jt}^{\alpha_t - 1} \cdot \prod_n \prod_t \theta_{jt}^{z_{jnt}} = Dir\left(\bar\theta_j \mid \bar\alpha + \sum_{n=1}^{N_j} \bar z_{jn}\right)$$

2) $\bar\varphi_t \sim p(\bar\varphi_t \mid \Phi_{-t}, \Theta, Z, W, \bar\alpha, \bar\beta) \propto$

$$\propto Dir(\bar\varphi_t \mid \bar\beta) \cdot \prod_{j=1}^{M} \prod_{n=1}^{N_j} Mult(\bar w_{jn} \mid \varphi_{\bar z_{jn}}) \propto$$

$$\propto Dir(\bar\varphi_t \mid \bar\beta) \prod_j \prod_n Mult(\underbrace{\bar w_{jn}}_{=(0 \ldots 1 \ldots 0)} \mid \bar\varphi_t)^{[z_{jn} = t]}$$

$$= Dir\left(\bar\varphi_t \mid \bar\beta + \sum_{j=1}^{M} \sum_{n=1}^{N_j} [z_{jn} = t] \cdot \bar w_{jn}\right)$$

3) $\bar{z}_{jn} \sim p(\bar{z}_{jn} | z_{-jn}, \Theta, \Phi, \bar{\alpha}, \bar{\beta})$

$$p(z_{jnt}=1 | \bar{\theta}_j, \Phi, \bar{w}_{jn}) = \frac{p(z_{jnt}=1, \bar{w}_{jn} | \bar{\theta}_j, \Phi)}{p(\bar{w}_{jn} | \bar{\theta}_j, \Phi)} =$$

$$= \frac{\theta_{jt}\,\varphi_{t,w_{jn}}}{\sum_s \theta_{js}\,\varphi_{s,w_{jn}}}$$

② Collapsed Gibbs sampling

$p(\underbrace{\Theta, \Phi}, z | W, \bar{\alpha}, \bar{\beta})$ — $\bar{\theta}_1 - \bar{\theta}_M, \bar{\varphi}_1 - \bar{\varphi}_T, \bar{z}_{11} - \bar{z}_{M,N_M}$

$$\bar{z}_{jn} \sim p(\bar{z}_{jn} | z_{-jn}, \underset{\Theta}{W}, \bar{\alpha}, \bar{\beta}) \underset{\Phi}{\propto}$$

$$\propto p(z, W | \bar{\alpha}, \bar{\beta}) = p(z|\bar{\alpha})\, p(W | z, \bar{\beta}).$$

$$p(z|\bar{\alpha}) = \int p(z, \Theta | \bar{\alpha})\, d\Theta = \int p(z|\Theta)\, p(\Theta|\bar{\alpha})\, d\Theta =$$

$$= \int \left(\prod_j p(\bar{\theta}_j | \bar{\alpha})\right)\left(\prod_{j=1}^{M} \prod_{n=1}^{N_j} p(\bar{z}_{jn} | \bar{\theta}_j)\right) d\Theta =$$

$$= \int \left(\prod_{j=1}^{M} \frac{1}{B(\bar{\alpha})} \prod_t \theta_{jt}^{\alpha_t - 1}\right)\left(\prod_j \prod_n \prod_t \theta_{jt}^{z_{jnt}}\right) d\Theta =$$

$$= \frac{1}{B(\bar{\alpha})^M} \int \prod_{j=1}^{M} \underbrace{\left(\prod_{t=1}^{T} \theta_{jt}^{\alpha_t + \sum_n z_{jnt} - 1}\right)}_{\text{“} \bar{\theta}_j} d\Theta = \quad = B(\bar{\alpha} + \sum_n \bar{z}_{jn})$$

$$= \prod_{j=1}^{M} \frac{B(\bar{\alpha} + \sum_n \bar{z}_{jn})}{B(\bar{\alpha})} = p(z|\bar{\alpha})$$

$$p(W \mid Z, \bar\beta) = \int p(W \mid Z, \Phi)\, p(\Phi \mid \bar\beta)\, d\Phi =$$

$$= \int \left( \prod_{t=1}^{T} \frac{1}{B(\bar\beta)} \cdot \prod_{v=1}^{V} \varphi_{tv}^{\beta_v - 1} \right) \cdot \left( \underbrace{\prod_{j=1}^{M} \prod_{n=1}^{N_j}} \prod_{t=1}^{T} \prod_{v=1}^{V} \varphi_{tv}^{z_{jnt} \cdot [w_{jn} = v]} \right) d\Phi$$

$$= \frac{1}{B(\bar\beta)^T} \cdot \int \prod_{t=1}^{T} \left( \prod_{v=1}^{V} \varphi_{tv}^{\beta_v + \sum_j \sum_n z_{jnt} \cdot [w_{jn} = v] - 1} \right) d\Phi =$$

$$= \prod_{t=1}^{T} \frac{B(\bar\beta + \bar\delta_t)}{B(\bar\beta)}$$ где $\delta_{tv} = \sum_j \sum_n z_{jnt}[w_{jn} = v]$

$$p(Z, W \mid \bar\alpha, \bar\beta) = \left[ \prod_{j=1}^{M} \frac{B(\bar\alpha + \bar\sigma_j)}{B(\bar\alpha)} \right] \times \left[ \prod_{t=1}^{T} \frac{B(\bar\beta + \bar\delta_t)}{B(\bar\beta)} \right]$$

$$p(z_{jnt} = 1 \mid Z_{-jn}, W, \bar\alpha, \beta) \propto p(Z, W \mid \bar\alpha, \bar\beta) \propto$$

$$\propto \frac{B(\bar\alpha + \bar\sigma_j)}{B(\bar\alpha)} \cdot \prod_{s=1}^{T} \frac{B(\bar\beta + \bar\delta_s)}{B(\bar\beta)} \propto B(\bar\alpha + \bar\sigma_j) \prod_s B(\bar\beta + \bar\delta_s)$$

$$= \left( \frac{\prod_s \Gamma(\alpha_s + \sigma_{js})}{\underbrace{\Gamma\left(\sum_s (\alpha_s + \sigma_{js})\right)}_{= \sum_s (\alpha_s + \sigma_{js}^{-jn}) + 1}} \right) \prod_s \frac{\prod_{v=1}^{V} \Gamma(\beta_v + \delta_{sv})}{\Gamma\left(\sum_v (\beta_v + \delta_{sv})\right)} =$$

$$\sigma_{is}^{-jn} = \begin{cases} \sigma_{is} - 1, & \text{если } i = j,\ s = z_{jn} \\ \sigma_{is}, & \text{в пр. сл.} \end{cases}$$

$= \sum_m z_{im}$ кроме $z_{jn}$

не зав. от $z_{jn}$, только от $z_{-jn}$

$$\prod_s \Gamma(\alpha_s+\sigma_{js}) = \underbrace{\prod_{s\neq t}\Gamma(\alpha_s+\sigma_{js}^{-jn})\cdot \underbrace{\Gamma(\alpha_t+\sigma_{jt}^{-jn}+1)}_{\text{"}\Gamma(\cdot)(\cdot)}} =$$

$$= \prod_{s=1}^{T}\Gamma(\alpha_s+\sigma_{js}^{-jn})\cdot(\alpha_t+\sigma_{jt}^{-jn})$$

$$= \frac{\overset{\text{const}}{\cancel{\prod_{s=1}^{T}\Gamma(\alpha_s+\sigma_{js}^{-jn})}}\cdot(\alpha_t+\sigma_{jt}^{-jn})}{\cancel{\Gamma\left(\sum_s(\alpha_s+\sigma_{js}^{-jn})\right)}\cdot\underset{\text{const}}{\cancel{\left(\sum_s(\alpha_s+\sigma_{js}^{-jn})\right)}}} \quad \times$$

$$\delta_{sv}^{-jn} = \delta_{sv} - [z_{jn}=s, w_{jn}=v]$$

$$\prod_s\prod_v\Gamma(\beta_v+\delta_{sv}) = \prod_s\prod_v\Gamma(\beta_v+\delta_{sv}^{-jn})\cdot(\beta_{w_{jn}}+\delta_{z_{jn},w_{jn}})$$

$$\prod_s\Gamma\left(\sum_v(\beta_v+\delta_{sv}^{-jn})+[z_{jn}=s]\right) =$$

$$= \prod_s\Gamma\left(\sum_v(\beta_v+\delta_{sv}^{-jn})\right)\cdot\left(\sum_v(\beta_v+\delta_{z_{jn},v}^{-jn})\right)$$

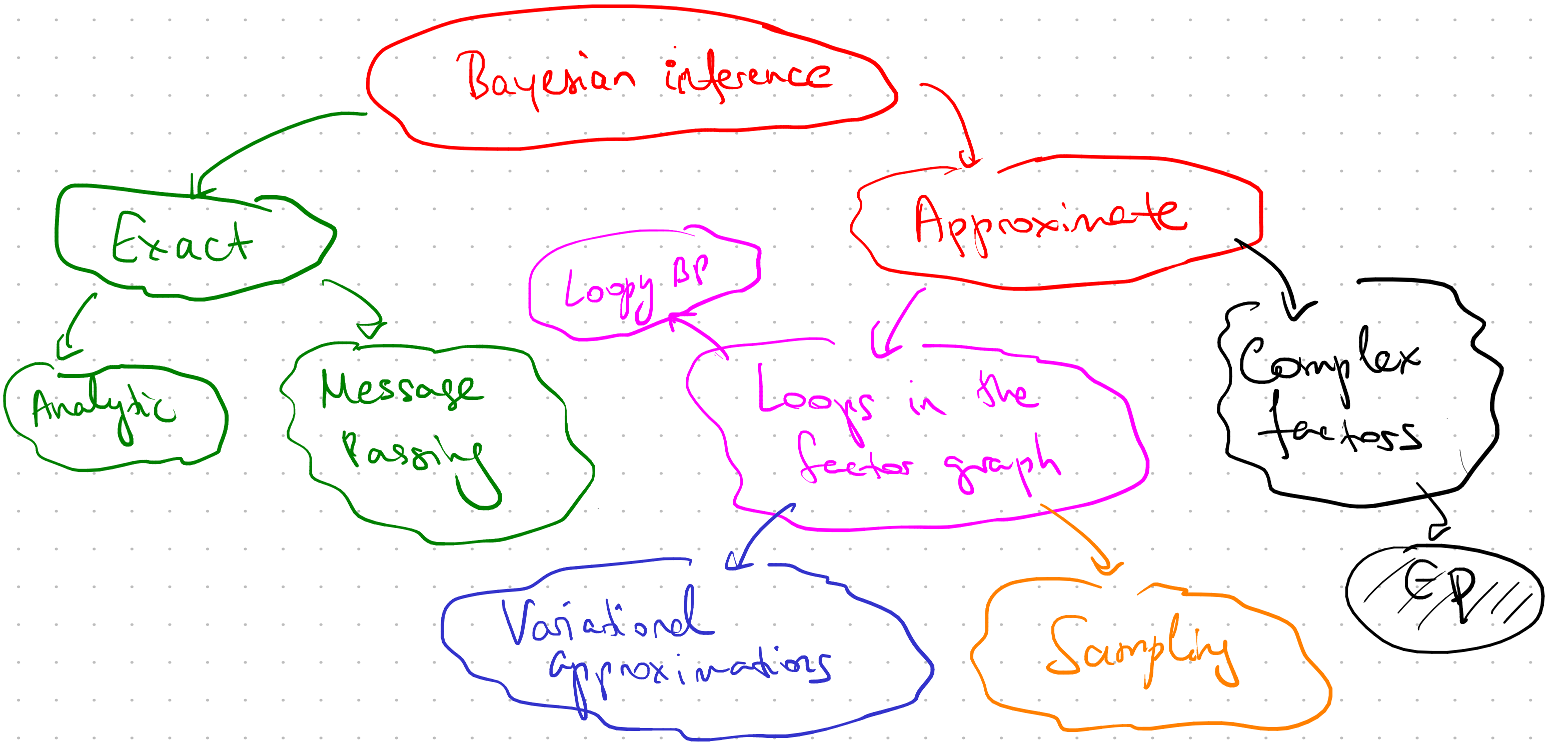
$$\times \quad \frac{\overset{\text{const}}{\cancel{\prod_s\prod_v\Gamma(\beta_v+\delta_{sv}^{-jn})}}}{\cancel{\prod_s\Gamma\left(\sum_v(\beta_v+\delta_{sv}^{-jn})\right)}}\cdot\frac{\beta_{w_{jn}}+\delta_{t,w_{jn}}^{-jn}}{\sum_v(\beta_v+\delta_{tv}^{-jn})}$$

$$p(z_{jn}=t\mid z_{-jn},W,\alpha,\beta)\propto(\sigma_{jt}^{-jn}+\alpha_t)\cdot\underbrace{\frac{\beta_{w_{jn}}+\delta_{t,w_{jn}}^{-jn}}{\sum_v(\beta_v+\delta_{tv}^{-jn})}}$$

$$\theta_{jt} = \frac{\alpha_t + \sum_{n=1}^{N_j} z_{jnt}}{\sum_s \left(\alpha_s + \sum_n z_{jns}\right)} \qquad \varphi_{ts} = \frac{\beta_s + \sum_j \sum_n z_{jnt}[w_{jn} = s]}{\sum_{s'} \beta_{s'} + \sum_j \sum_n z_{jnt}}$$

## Neural topic modeling

Text $\rightarrow$ RoBERTa $\rightarrow$ $\bar{x} \in \mathbb{R}^d$




## (3) Moment matching

$q(\bar{x}) \approx p(\bar{x})$ $\qquad KL(p\|q) \xrightarrow[\bar{\eta}]{} \min$

$q = h(\bar{x}) \cdot e^{-\bar{\eta}^T \bar{u}(\bar{x}) + a(\bar{\eta})}$

$$KL(p\|q) = \text{const} - \int p(\bar{x}) \left( \log h(\bar{x}) - \bar{\eta}^T \bar{u}(\bar{x}) + a(\bar{\eta}) \right) d\bar{x} =$$

$\int p \log \frac{p}{q}$

$$= \text{Const} + \bar{\eta}^T \int p(\bar{x}) \bar{u}(\bar{x}) d\bar{x} - a(\bar{\eta}) \xrightarrow[\bar{\eta}]{} \min$$

$= E_{q(\bar{x})}[\bar{u}(\bar{x})]$

$$\nabla_{\bar{\eta}} KL = E_{p(\bar{x})}[\bar{u}(\bar{x})] - \nabla_{\bar{\eta}} a(\bar{\eta}) = 0$$

Moment matching

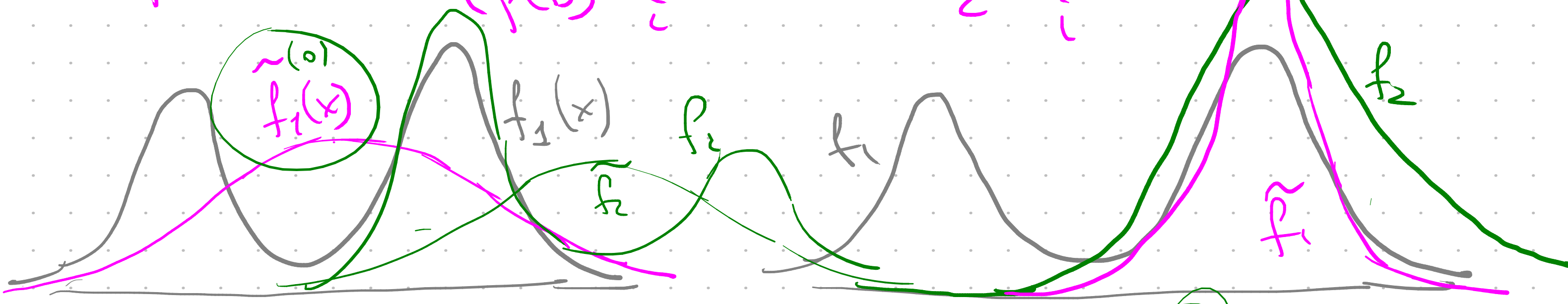
$$KL(p\|q) \xrightarrow{\bar q} \min \iff E_{q(\bar x)}[\underbrace{\bar u(\bar x)}_{\binom{\tau}{\mu\tau}}] = E_{p(\bar x)}[\bar u(\bar x)]$$

## 4 Expectation Propagation

$$p(\bar\theta|D) = \frac{1}{D}\, p(\bar\theta)\, p(D|\bar\theta) = \frac{1}{D}\, p(\bar\theta) \prod_n p(d_n|\bar\theta)$$

$$p(\bar\theta|D) \propto \prod_i f_i(\bar\theta_i) \approx q(\bar\theta) = \frac{1}{Z} \prod_i \tilde f_i(\bar\theta_i)$$

$$KL(p\|q) = KL\left(\frac{1}{p(D)} \prod_i f_i(\bar\theta) \,\Big\|\, \frac{1}{Z} \prod_i \tilde f_i(\bar\theta)\right) \xrightarrow{\tilde f_i} \min$$




$\tilde f_1 \approx f_1$ $\quad$ $\cancel{\tilde f_1 \approx f_1}$ $\quad$ $\tilde f_1 \cdot f_2 \approx f_1 \cdot f_2$

$\tilde f_1 \cdot \tilde f_2 \approx f_1 \cdot \tilde f_2$

### Expectation Propagation:

- init $\tilde f_j^{(0)}(\bar\theta) \approx f_j(\bar\theta)$
- for $j = 1..n, 1, .., n, ....$; $\quad k = 1, 2,$ —
  - find $\tilde f_j^{(k+1)}(\bar\theta)$:

$$\tilde f_j^{(k+1)}(\bar\theta) = \arg\min_{\tilde f_j} KL\left(\frac{1}{Z_j} f_j(\bar\theta) \cdot \prod_{i\neq j} \tilde f_i^{(k)}(\bar\theta) \,\Big\|\, \overbrace{\tilde f_j(\bar\theta) \cdot \prod_{i\neq j} \tilde f_i^{(k)}(\bar\theta)}^{=\, q(\bar\theta)}\right)$$

1) $q^{-j}(\bar\theta) = \prod_{i\neq j} \tilde f_i(\bar\theta) = \dfrac{q(\bar\theta)}{\tilde f_j(\bar\theta)}$

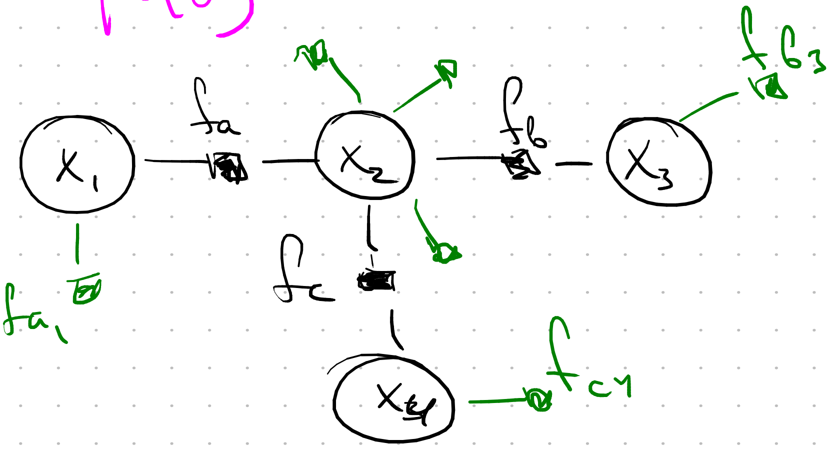
$= \tilde f_j(\bar\theta) \cdot q^{-j}(\bar\theta)$

2) $q^{new}(\bar\theta) = \arg\min KL\left(\frac{1}{Z_j} f_j(\bar\theta)\cdot q^{-j}(\bar\theta) \,\|\, q^{new}(\bar\theta)\right)$

(moment matching)

3) $\tilde f_j(\bar\theta) = \dfrac{q^{new}(\bar\theta)}{q^{-j}(\bar\theta)}$

Пример.




$$p(x_1 - x_4) = f_a(x_1, x_2)\, f_b(x_2, x_3)\, f_c(x_2, x_4)$$

$$\approx q(x_1 - x_4) = \tilde f_a(x_1, x_2)\, \tilde f_b(x_2, x_3)\, \tilde f_c(x_2, x_4) =$$

$$= \tilde f_{a_1}(x_1)\, \tilde f_{a2}(x_2)\, \tilde f_{b2}(x_2)\, \tilde f_{b3}(x_3)\, \tilde f_{c2}(x_2)\, \tilde f_{c4}(x_4)$$

$\tilde f_b(x_2, x_3) = \tilde f_{b2} \cdot \tilde f_{b3}$

$KL(p \,\|\, q = \prod q_i)$

$\Leftrightarrow$

$q_i = \int p(\bar x)\, d\bar x_{-i}$

$q^{-b}(x_1 - x_4) = \tilde f_{a1} \cdot \tilde f_{a2} \cdot \tilde f_{c2} \cdot \tilde f_{c4}$

$q^{new} = \arg\min KL\left(f_b(x_2, x_3) \cdot q^{-b} \,\|\, q^{new}(\bar x)\right)$

$\|$

$q_1^{new}(x_1)\, q_2^{new} \ldots q_4^{new}(x_4)$

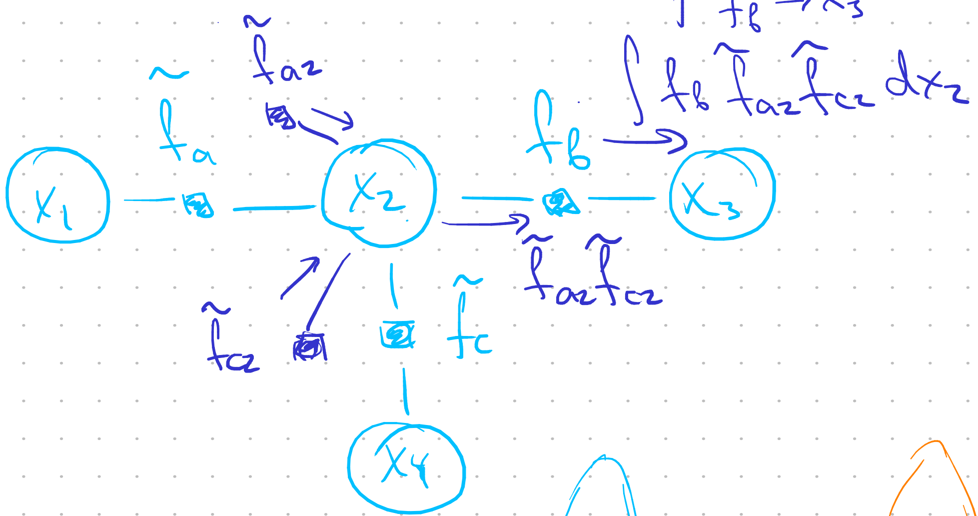
$q_1^{new}(x_1) = \int f_b(x_2, x_3)\, \tilde f_{a1} \tilde f_{a2} \tilde f_{c2} \tilde f_{c4}\, dx_2 dx_3 dx_4 \propto \tilde f_{a1}(\bar x_1)$

$q_4^{new}(x_4) = \tilde f_{c4}(\bar x_4)$

$$q_2^{new}(x_2) = \int f_b(x_2,x_3)\, \cancel{\tilde f_{a1}}\, \tilde f_{a2}\, \hat f_{c2}\, \cancel{\tilde f_{c4}}\, \cancel{dx_1}\, dx_3\, \cancel{dx_4} =$$

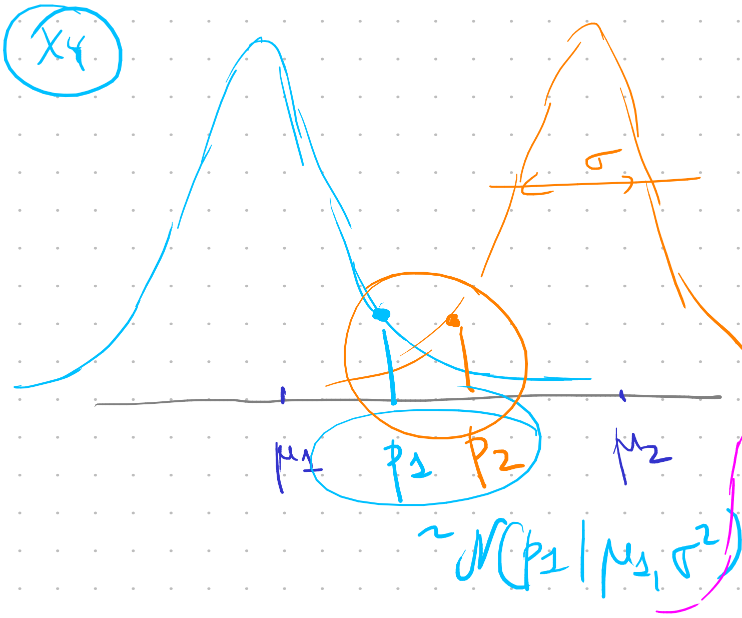
$$= \int f_b(x_2,x_3)\, \tilde f_{a2}(x_2)\, \hat f_{c2}(x_2)\, dx_3 = \tilde f_{a2} \hat f_{c2} \int f_b\, dx_3$$

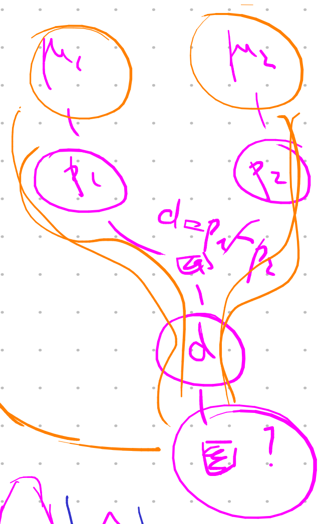
$$q_3^{new}(x_3) = \underbrace{\int f_b(x_2,x_3)\, \tilde f_{a2}(x_2)\, \hat f_{c2}(x_2)\, dx_2}_{=\ \mu_{f_b \to x_3}(x_3)}$$




③ TrueSkill

Elo rating




$$\begin{cases} p_1 > p_2 + \varepsilon & 1 \\ |p_1 - p_2| < \varepsilon & 1/2 \\ p_1 + \varepsilon < p_2 & 0 \end{cases}$$

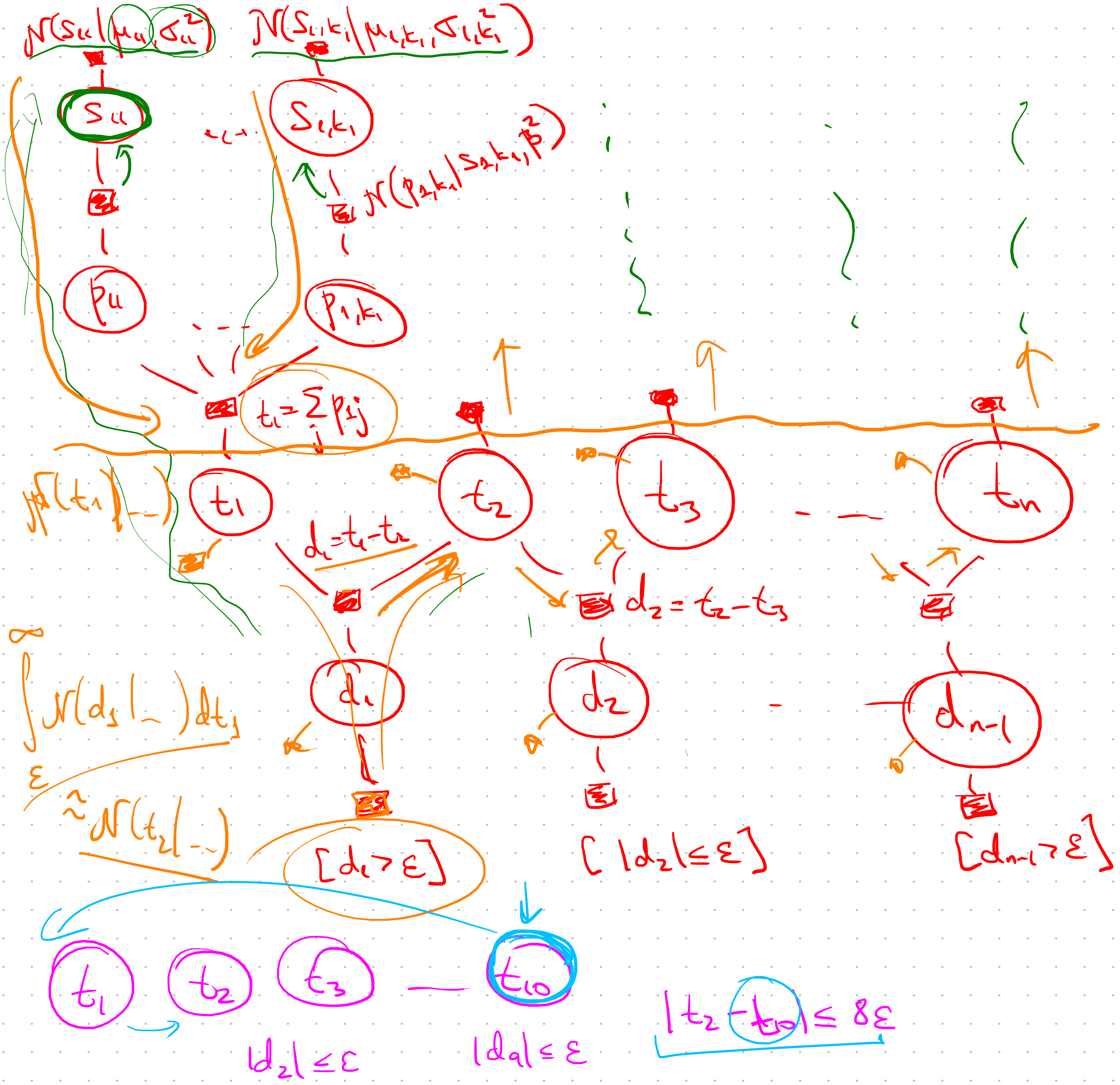

$$p(\mu_1 \mid D) \propto p(\mu_1) \cdot p(p_1 - p_2 \mid |p_1 - p_2| < \varepsilon)$$

$t_1$ : $s_{11} \dots s_{1,k_1}$

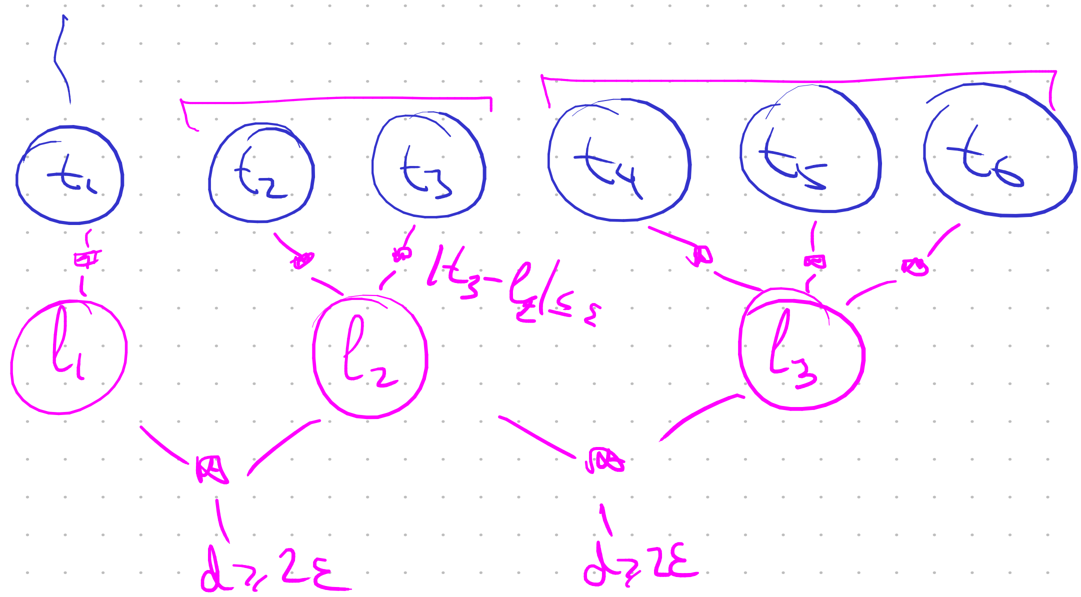
$t_2$

⋮

$t_n$ : $s_{n1} \dots s_{n,k_n}$



N(s11 | μ11, σ11²)
N(s1,k1 | μ1,k1, σ1,k1²)
s11
s1,k1
N(p1,k1 | s1,k1, β²)
p11
p1,k1
t1 = Σ p1j
t1
t2
t3
tn
N(t1 | _)
d1 = t1 − t2
d2 = t2 − t3
d1
d2
dn−1
∫ε^∞ N(d1 | _) dt1
≈ N(t2 | _)
[d1 > ε]
[ |d2| ≤ ε ]
[dn−1 > ε]
t1
t2
t3
t10
|d2| ≤ ε
|d9| ≤ ε
|t2 − t10| ≤ 8ε

t1
t2
t3
t4
t5
t6
|t3 − l2| ≤ ε
l1
l2
l3
d ≥ 2ε
d ≥ 2ε