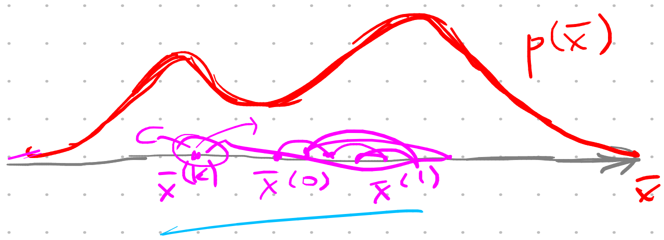
① Markov chains



$\bar{x}^{(0)}, \bar{x}^{(1)}, \bar{x}^{(2)}, \ldots$

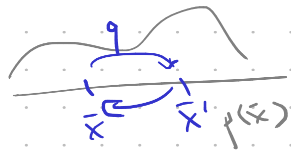
$q^{(0)}(\bar{x}), \; q^{(1)}(\bar{x}), \; \ldots, \; q^{(k)}(\bar{x}), \ldots \to \boxed{p(\bar{x})}$

$q(\bar{x}^{(t)} \mid \bar{x}^{(t-1)})$

$$q^{(t+1)}(\bar{x}) = \int q(\bar{x} \mid \bar{x}')\, q^{(t)}(\bar{x}')\, d\bar{x}'$$

$q^{(0)}, q^{(1)}, q^{(2)}, \ldots, q^{(t)}, \ldots \to \pi(\bar{x})$ — stationary distribution

$\pi(\bar{x})$: $\quad \pi(\bar{x}) = \int q(\bar{x} \mid \bar{x}')\, \pi(\bar{x}')\, d\bar{x}'$



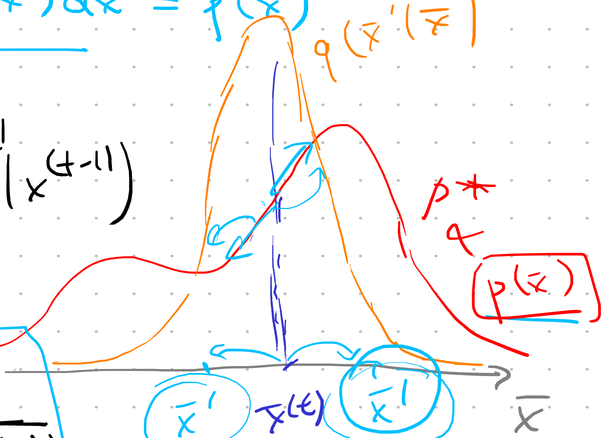
Условие баланса: если $\forall \bar{x}, \bar{x}'$ $\quad p(\bar{x})q(\bar{x}' \mid \bar{x}) = p(\bar{x}')q(\bar{x} \mid \bar{x}')$,

то $p(\bar{x})$ — это стацион. распред.

$$\int q(\bar{x} \mid \bar{x}')\, p(\bar{x}')\, d\bar{x}' = \int p(\bar{x})\, q(\bar{x}' \mid \bar{x})\, d\bar{x}' = p(\bar{x})$$

② Metropolis-Hastings algorithm

$\forall$ Markov chain $\quad \bar{x}^{(0)}, \bar{x}^{(1)}, \ldots \qquad q(\bar{x}^{(t)} \mid x^{(t-1)})$



Repeat:

- sample $\bar{x}' \sim q(\bar{x}' \mid \bar{x}^{(t)})$
- $a(\bar{x}', \bar{x}^{(t)}) = \boxed{\dfrac{p^*(\bar{x}')}{p^*(\bar{x}^{(t)})}} \boxed{\dfrac{q(\bar{x}^{(t)} \mid \bar{x}')}{q(\bar{x}' \mid \bar{x}^{(t)})}}$

$a(\bar{x}', \bar{x}) = \dfrac{1}{a(\bar{x}, \bar{x}')}$

- if $a \geq 1$ then accept $\bar{x}^{(t+1)} := \bar{x}'$

else $\quad \bar{x}^{(t+1)} := \begin{cases} \bar{x}^{(t)} & \text{с вероятн. } 1-a \\ \bar{x}' & \text{с вероятн. } a \end{cases}$

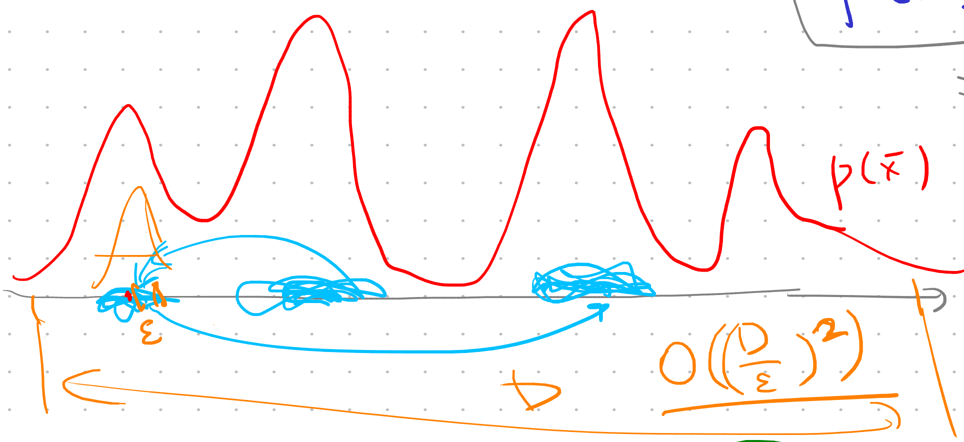
$$\forall \bar{x}, \bar{x}' \qquad p(\bar{x}) \cdot p_{MC}(\bar{x}' \mid \bar{x}) \overset{?}{=} p(\bar{x}')\, p_{MC}(\bar{x} \mid \bar{x}')$$

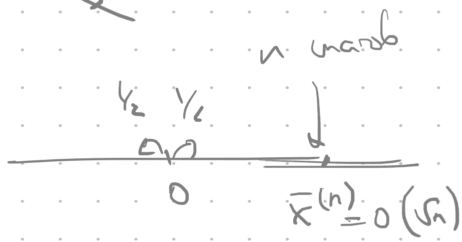
Пусть $a(\bar{x}', \bar{x}) \geq 1$: $\quad p_{MC}(\bar{x}' \mid \bar{x}) = q(\bar{x}' \mid \bar{x})$

$\qquad p_{MC}(\bar{x} \mid \bar{x}') = q(\bar{x} \mid \bar{x}') \cdot a(\bar{x}, \bar{x}')$

$$p(\bar x)\, q(\bar x' | \bar x) \overset{?}{=} \cancel{p(\bar x')}\, \cancel{q(\bar x | \bar x')} \cdot \boxed{\frac{p^*(\bar x)}{p^*(\bar x')}} \cdot \frac{q(\bar x' | \bar x)}{\cancel{q(\bar x | \bar x')}}$$

$$= \frac{p(\bar x)}{\cancel{p(\bar x')}}$$





$\mathcal{N}(\bar x | \hat{\mu}, \Sigma)$ $\quad \bar\mu \sim p(\bar\mu | D) \propto p(\bar\mu) \cdot p(D | \bar\mu)$

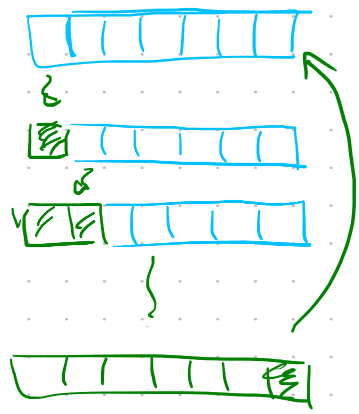
$$\bar x \sim p(\bar x | D) = \int p(\bar x | \bar\mu)\, p(\bar\mu | D)\, d\bar\mu \approx \frac{1}{R} \sum_r p(\bar x | \bar\mu^{(r)})$$

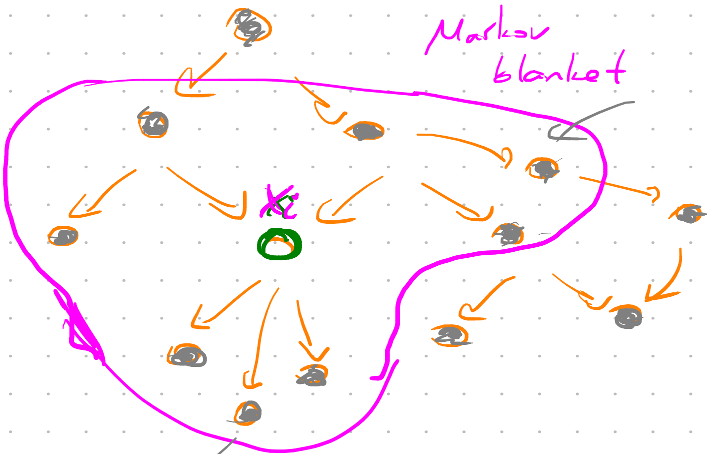
③ Gibbs sampling

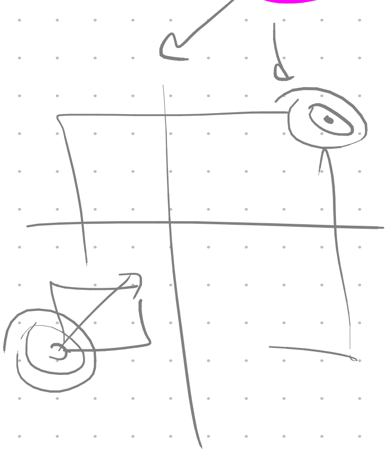
$$p(\bar x) = p(x_1, x_2, \dots, x_n)$$

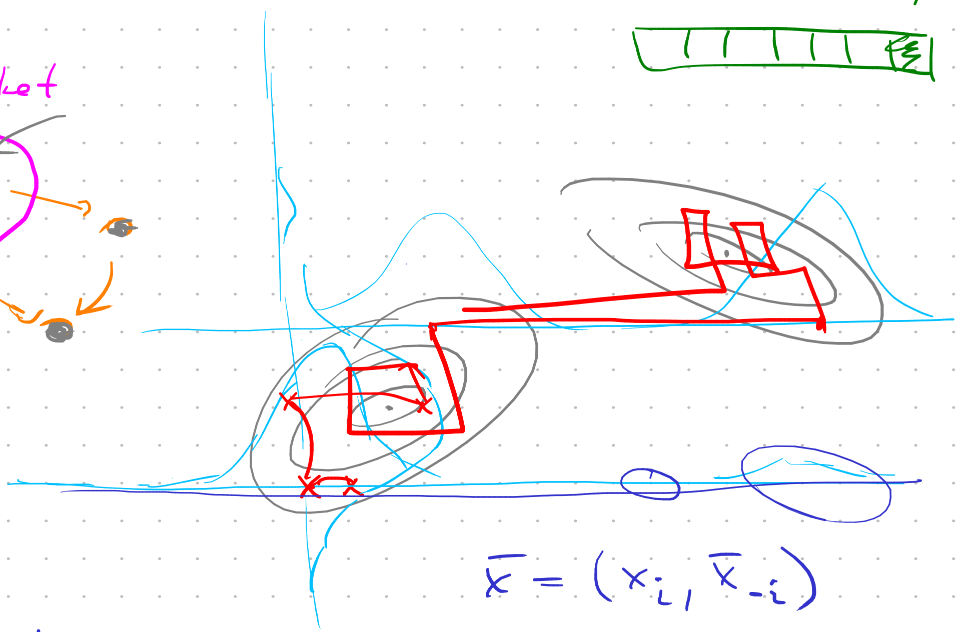
- init $\bar x^{(0)}$
- repeat:
  - for $i = 1..n$:

$$x_i^{(t+1)} \sim p(x_i \mid x_1^{(t+1)}, \dots, x_{i-1}^{(t+1)}, x_{i+1}^{(t)}, \dots, x_n^{(t)})$$









$$\bar x = (x_i, \bar x_{-i})$$

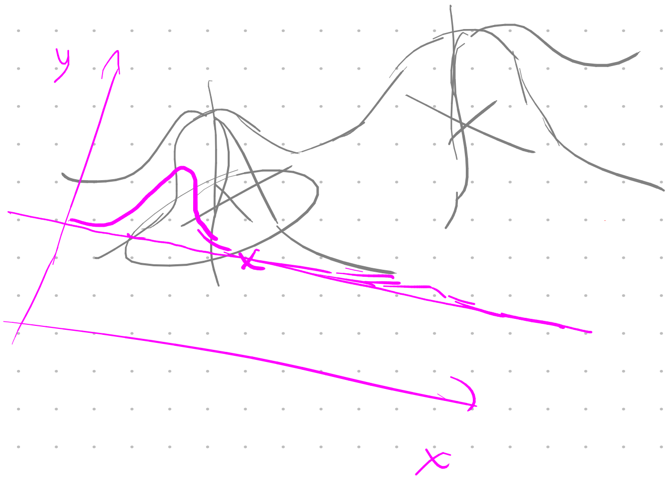
– i-й вар!

$$q(\bar x' | \bar x) = \begin{cases} 0, & \bar x'_{-i} \neq \bar x_{-i} \\ p(x'_i | \bar x_{-i}), & \bar x'_{-i} = \bar x_{-i} \end{cases}$$

$$a(\bar{x}', \bar{x}) = \frac{p^*(\bar{x}')}{p^*(\bar{x})} \cdot \frac{q(\bar{x} \mid \bar{x}')}{q(\bar{x}' \mid \bar{x})} = \frac{p(x_i', \bar{x}_{-i}')}{p(x_i, \bar{x}_{-i})} \cdot \frac{p(x_i \mid \bar{x}_{-i}')}{p(x_i' \mid \bar{x}_{-i})} =$$

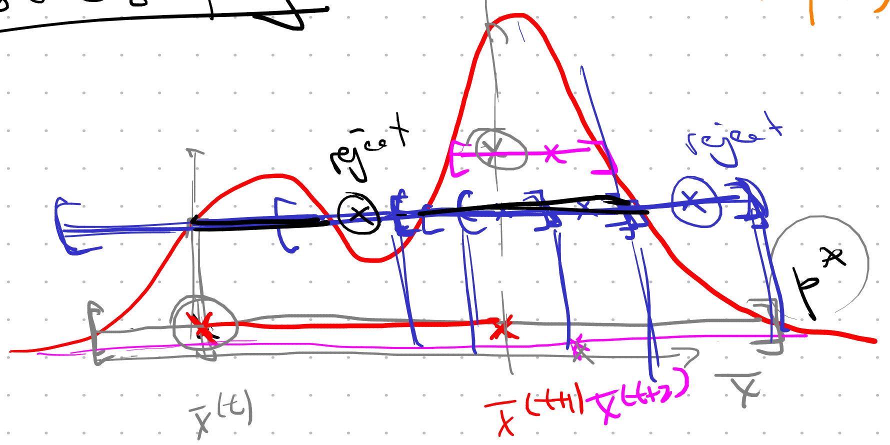
$$\bar{x}_{-i}' = \bar{x}_{-i}$$

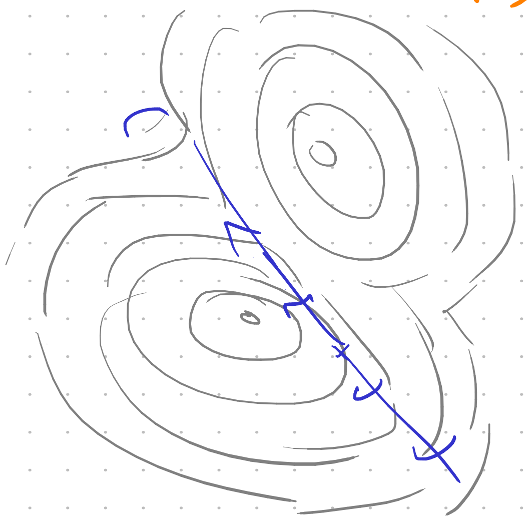
$$= \frac{p(\bar{x}_{-i})\, p(x_i' \mid \bar{x}_{-i})}{p(\bar{x}_{-i})\, p(x_i \mid \bar{x}_{-i})} \cdot \frac{p(x_i \mid \bar{x}_{-i})}{p(x_i' \mid \bar{x}_{-i})} = 1$$



4 Slice sampling

$$\bar{x} \sim p(\bar{x}) \Leftrightarrow (\bar{x}, y) \sim \mathrm{Unif}(\text{под } p^*)$$





5 SIR models

$S \to I \to R$

susceptible — infected — recovered / removed

$$X = \{x_1, x_2, \dots, x_N\}$$

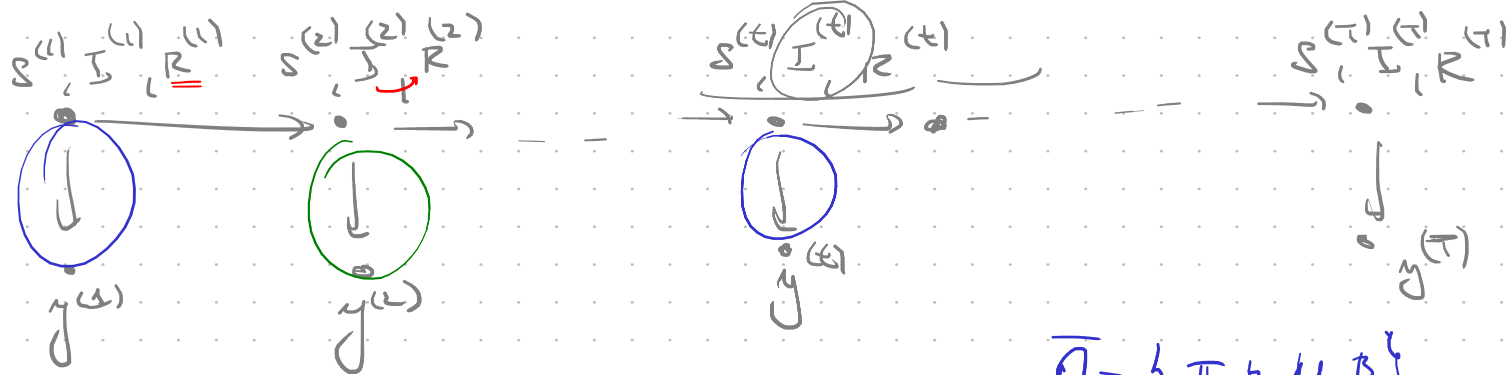
$$x_i^{(t)} \in \{S, I, R\}$$

$$y = \{y^{(1)}, y^{(2)}, \dots, y^{(t)}\}$$

\# выявл заболевших во время t

$$S^{(t)} = \#\{i \mid x_i^{(t)} = S\}$$

$$I^{(t)}, R^{(t)} \qquad \forall t \quad S^{(t)} + I^{(t)} + R^{(t)} = N$$



$$\bar{\theta} = \{\pi, p, \mu, \beta\}$$

Параметры модели:

1) $R^{(1)} = 0$, $p(x_i^{(1)} = I) = \pi$, $p(x_i^{(1)} = S) = 1 - \pi$ $\qquad \pi$

2) $p(x_i^{(t)} \in y^{(t)} \mid x_i^{(t)} = I) = p$

$$p(y^{(t)} \mid I^{(t)}, p) = \mathrm{Binom}(y^{(t)} \mid I^{(t)}, p) \qquad p$$

3) $p(x_i^{(t+1)} = R \mid x_i^{(t)} = I) = \mu$ $\qquad \mu$

— " — I — " — $= 1 - \mu$

4) p(заразиться от 1 контакта) $= \beta$ $\qquad \beta$

$$p(x_i^{(t+1)} = S \mid x_i^{(t)} = S) = (1-\beta)^{I^{(t)}}$$

— " — I — " — $= 1 - (1-\beta)^{I^{(t)}}$

$$p(x_i^{(t+1)} \mid x_i^{(t)}) = \begin{array}{c|ccc} & S & I & R \\ \hline S & (1-\beta)^{I^{(t)}} & 1-(1-\beta)^{I^{(t)}} & 0 \\ I & 0 & 1-\mu & \mu \\ R & 0 & 0 & 1 \end{array}$$

6 Inference in SIR

$$p(X, y \mid \bar{\theta}) = p(X^{(1)} \mid \pi)\, p(y^{(1)} \mid X^{(1)}, p)\, p(X^{(2)} \mid X^{(1)}, \beta, \mu) \ldots p(y^{(T)} \mid X^{(T)}, p)$$

$$= \left[\pi^{I^{(1)}} (1-\pi)^{S^{(1)}}\right] \left[\binom{I^{(1)}}{y^{(1)}} p^{y^{(1)}} (1-p)^{I^{(1)} - y^{(1)}}\right] \left[\ldots\right] \ldots =$$

$$= \pi^{I^{(1)}} (1-\pi)^{S^{(1)}} \cdot \left[\prod_{t=1}^{T} \overset{\text{const}}{\cancel{\binom{I^{(t)}}{y^{(t)}}}}\, p^{y^{(t)}} (1-p)^{I^{(t)} - y^{(t)}}\right] \cdot \left[\prod_{t=1}^{T} \prod_{j=1}^{N} p(x_j^{(t+1)} \mid x_j^{(t)}, \mu, \beta, I_{-j}^{(t)})\right]$$

$$p(y|\bar\theta) = \int p(y|x,\bar\theta)\,p(x|\bar\theta)\,dx = \mathbb{E}_{p(x|\bar\theta)}\left[p(y|x,\bar\theta)\right] \approx \frac{1}{R}\sum_{r=1}^{R} p(y|x_r,\bar\theta)$$

sampling

$\bar\theta \to \max$

$$X = (\bar x_1, \bar x_2, \dots, \bar x_j, \dots, \bar x_N)$$

$$\bar x_j = (x_j^{(1)}, x_j^{(2)}, \dots, x_j^{(T)})$$

Gibbs sampling:

$1 \dots j \dots N$:

$$\boxed{\bar x_j \sim p(\bar x_j | X_{-j}, y, \bar\theta)}$$

EM

$$\begin{tikzcd}
\bar\theta_0 \arrow[r, "\text{Gibbs}"] & X_{0,r} \arrow[dl, "\text{MAP}"'] \\
\bar\theta_1 \arrow[r, "\text{Gibbs}"] & X_{1,r} \arrow[dl, "\text{MAP}"'] \\
\bar\theta_2 \arrow[r] & \cdots
\end{tikzcd}$$

7 Stochastic Viterbi algorithm

$$\bar x_j \sim p(\bar x_j | X_{-j}, y, \bar\theta)$$

$$\begin{tikzcd}
x_j^{(1)} \arrow[r] \arrow[d] & x_j^{(2)} \arrow[r] \arrow[d] & \cdots \arrow[r] & x_j^{(t-1)} \arrow[r] \arrow[d] & x_j^{(t)} \arrow[r] \arrow[d] & \cdots \arrow[r] & x_j^{(T)} \arrow[d] \\
y^{(1)} & y^{(2)} & & y^{(t-1)} & y^{(t)} & & y^{(T)}
\end{tikzcd}$$

$a, a' \in \{S, I, R\}$

$$\boxed{q^{(t)}_{j,a,a'}} = p\left(x_j^{(t)} = a, x_j^{(t+1)} = a' \,\middle|\, y^{(1)} \dots y^{(t)}, X_{-j}, \bar\theta\right)$$

$$q_j^{(t)} = \begin{array}{c|ccc} a \backslash a' & S & I & R \\ \hline S & & & \\ I & & & \\ R & & & \end{array}$$

$$\begin{tikzcd}
x_j^{(t-1)} \arrow[r] \arrow[d] & x_j^{(t)} \arrow[r] \arrow[d] & x_j^{(t+1)} \arrow[r] \arrow[d] & \cdots \\
y^{(t-1)} & y^{(t)} & y^{(t+1)} & \cdots
\end{tikzcd}$$

$$x_j^{(t)} \sim p\left(x_j^{(t)} = a \,\middle|\, x_j^{(t+1)} = a', y, X_{-j}, \bar\theta\right) =$$

$$= p\left(x_j^{(t)} = a \,\middle|\, x_j^{(t+1)} = a', y^{(1)} \dots y^{(t)}, X_{-j}, \bar\theta\right)$$

$$\underline{\underline{q^{(t)}_{j,a,a'}}} = p(x_j^{(t)} = a,\ x_j^{(t+1)} = a' \mid y^{(1)} - y^{(t)}, -) \propto$$

$$\propto p(x_j^{(t)} = a,\ x_j^{(t+1)} = a',\ y^{(t)} \mid y^{(1)} - y^{(t-1)}, -) =$$

$$= \underbrace{p(x_j^{(t)} = a \mid y^{(1)} - y^{(t-1)})}_{\sum_{a''} p(x_j^{(t)} = a,\ x_j^{(t-1)} = a'' \mid -)} \cdot \underbrace{p(y^{(t)} \mid x_j^{(t)} = a, -)}_{\mathrm{Binom}(y^{(t)} \mid I^{(t)}_{-j} + [a = I], \beta)} \cdot \underbrace{p(x_j^{(t+1)} = a' \mid x_j^{(t)} = a, -)}$$

$$\sum_{a''} p(x_j^{(t)} = a,\ x_j^{(t-1)} = a'' \mid -)$$

$$= \sum_{a''} q^{(t-1)}_{j,a'',a}$$

$$\begin{array}{c|ccc} & S\ I^{(t)}_{-j} & \overset{a'}{I} & R \\ \hline S & 1-\beta & - & 0 \\ \underset{I}{a} & 0 & 1-\mu & \mu \\ & 0 & 0 & 1 \end{array}$$

Sampling:

- $Q_j^{(t)} = a\begin{pmatrix} & a' & \\ - & q^{(t)}_{j,a,a'} & - \\ & | & \end{pmatrix}$ $\quad Q_j^{(T)}$

- sample $x_j^{(T)}, x_j^{(T-1)}, \ldots, x_j^{(1)}$:

$$x_j^{(T)} \sim p(x_j^{(T)} = a \mid y, X_{-j}, \theta) = \sum_{a'} q^{(T)}_{j,a,a'}$$

$$\ldots$$

$$x_j^{(t)} \sim p(x_j^{(t)} = a \mid x_j^{(t+1)} = a', y, X_{-j}, \bar\theta) \underline{\underline{\propto}} q_{j,a,a'}$$

$$\begin{array}{c|ccc} a \backslash a' & S & I & R \\ \hline S & & & \\ I & & & \\ R & & & \end{array}$$

- output $(x_j^{(1)} - x_j^{(T)})$

$\bar\theta = (\pi, p, \mu, \beta)$

(8) Learning

1) Init $\bar\theta$, X
2) for $j = 1, \ldots, N, 1, \ldots, N, \ldots$
   (i) $\bar{x}_j \sim$ Gibbs
   (ii) replace $\bar{x}_j$ to X
   (iii) every M steps update $\bar\theta$ from X

$$p(\bar\theta \mid X, y) \propto \underline{p(\bar\theta)}\, p(X, y \mid \bar\theta)$$

$$p(\pi) p(\mu) p(\beta) p(\phi)$$

$$\underline{p(\pi)} = \mathrm{Beta}(\pi \mid a_\pi, b_\pi) \propto \pi^{a_\pi - 1} (1-\pi)^{b_\pi - 1}$$

$$\log p(\bar\theta|X,y) = \text{Const} + \underline{(a_\pi-1)}\,\underline{\log\pi} + (b_\pi-1)\log(1-\pi) + \underline{I^{(1)}}\,\underline{\log\pi} + S^{(1)}\log(1-\pi) \quad \Big] \pi$$

$$+ (a_\rho-1)\log\rho + (b_\rho-1)\log(1-\rho) + \sum_{t=1}^{T}\left(y^{(t)}\log\rho + (I^{(t)}-y^{(t)})\log(1-\rho)\right) \quad \Big] \rho$$

$$+ (a_\mu-1)\log\mu + (b_\mu-1)\log(1-\mu) + \sum_{j=1}^{N}\sum_{t=1}^{T-1}\left([x_j^{(t)}=I, x_j^{(t+1)}=R]\log\mu + [x_j^{(t)}=I, x_j^{(t+1)}=I]\log(1-\mu)\right) \quad \Big] \mu$$

$$+ (a_\beta-1)\,\underline{\log\beta} + (b_\beta-1)\log(1-\beta) + \sum_{j=1}^{N}\sum_{t=1}^{T-1}\left([x_j^{(t)}=S, x_j^{(t+1)}=S]\cdot I^{(t)}\cdot\log(1-\beta) + \underline{[x_j^{(t)}=S, x_j^{(t+1)}=I]}\cdot ?\right) \quad \Big] \beta$$

$I^{(t)}$ контактов $\Rightarrow \geq 1$ раз заразился

$$? = \left(\underline{E[\#\text{ конт. с заражением}]}\cdot\log\beta + \underline{E[\#\text{ конт. без зараж.}]}\,\log(1-\beta)\right)$$

$$I^{(t)}\left(1-\frac{\beta}{1-(1-\beta)^{I^{(t)}}}\right)$$

$$I^{(t)}\cdot p\left(\begin{matrix}\text{заразился}\\ \text{при 1 конт.}\end{matrix}\,\Big|\,\text{зараз.}\right) = I^{(t)}\frac{\beta}{1-(1-\beta)^{I^{(t)}}}$$

$$p(\pi|\_\,\_) = \text{Beta}(a'_\pi, b'_\pi)$$

$$a'_\pi = a_\pi + I^{(1)} \qquad b'_\pi = b_\pi + S^{(1)}$$

$$a'_\rho = a_\rho + \sum_t y^{(t)} \qquad b'_\rho = b_\rho + \sum_t (I^{(t)}-y^{(t)})$$

$$a'_\mu = a_\mu + [\#\text{ перех. } I\to R] \qquad b'_\mu = b_\mu + [\#\text{ перех. } I\to I]$$

$$a'_\beta = a_\beta + \sum_{j=1}^{N}\sum_t [x_j^{(t)}=S, x_j^{(t+1)}=I]\cdot I^{(t)}\cdot\frac{\beta_{\text{п}}}{1-(1-\beta_{\text{п}})^{I^{(t)}}}$$

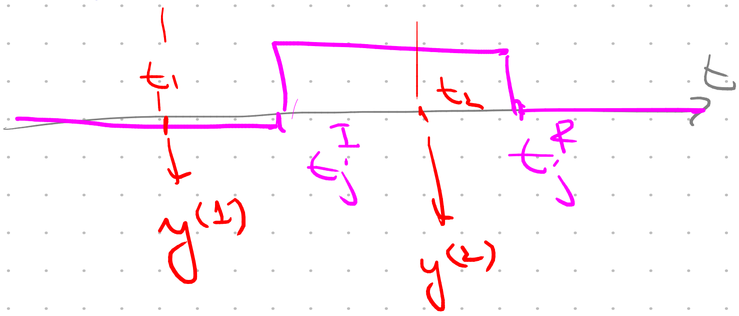
$$b'_\beta = b_\beta + \sum_{j=1}^{N} \sum_{t=1}^{T-1} [x_j^{(t)}=S]\left([x_j^{(t+1)}=S]\cdot I^{(t)} + [x_j^{(t+1)}=I]\cdot I^{(t)}\left(1-\frac{\beta_{14}}{1-(1-\beta_{14})^{I^{(t)}}}\right)\right)$$

9 Extensions

1) SIR $\longrightarrow$ SEIR (exposed) $\longrightarrow$ SEIRS

SIR $\longrightarrow$ SIRS $\longrightarrow$ SEIRS

2) Continious time



3) Модель заражения

$$p(S\,|\,S) = (1-\beta)^{|I^{(t)} \cap Nei(j)|}$$