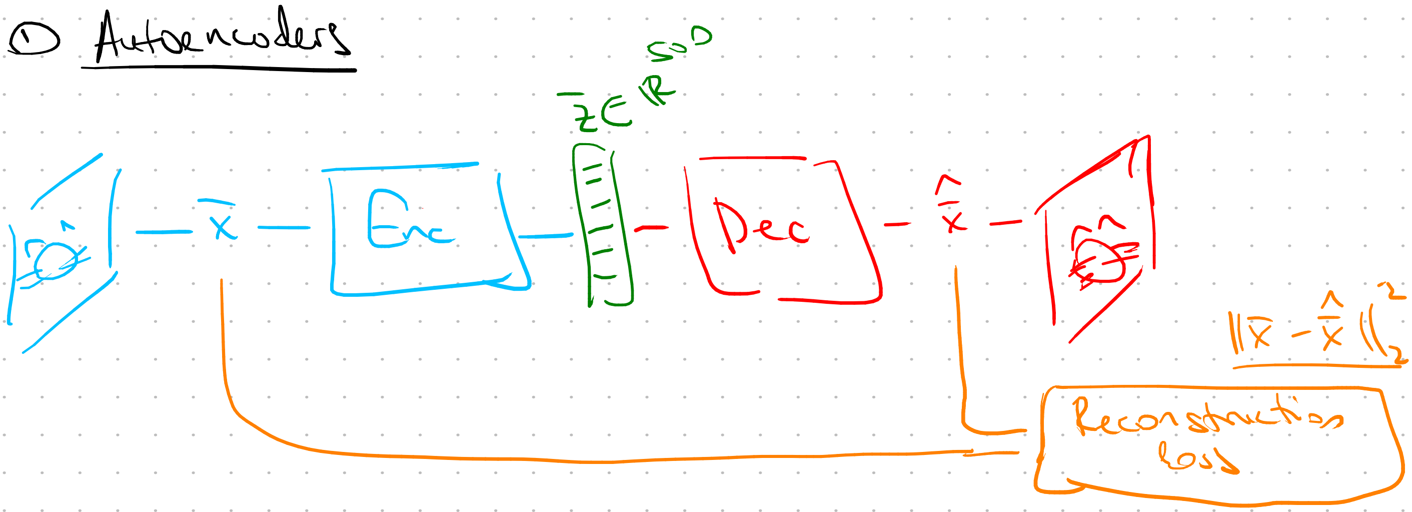
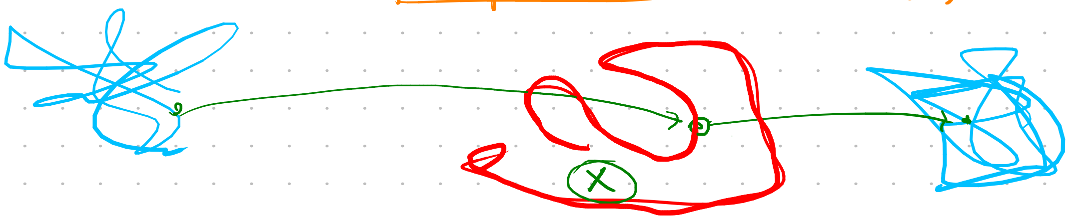
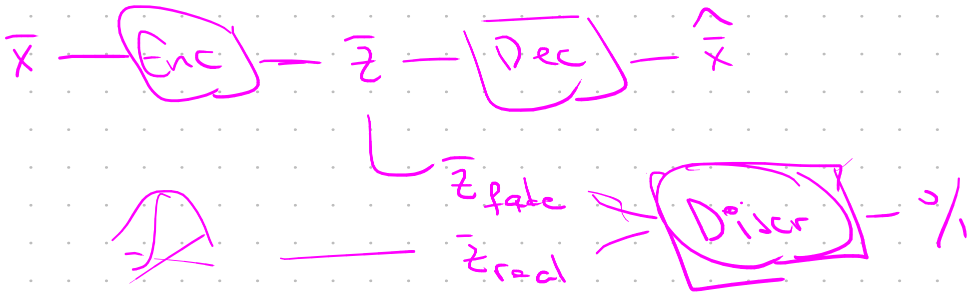
① Autoencoders



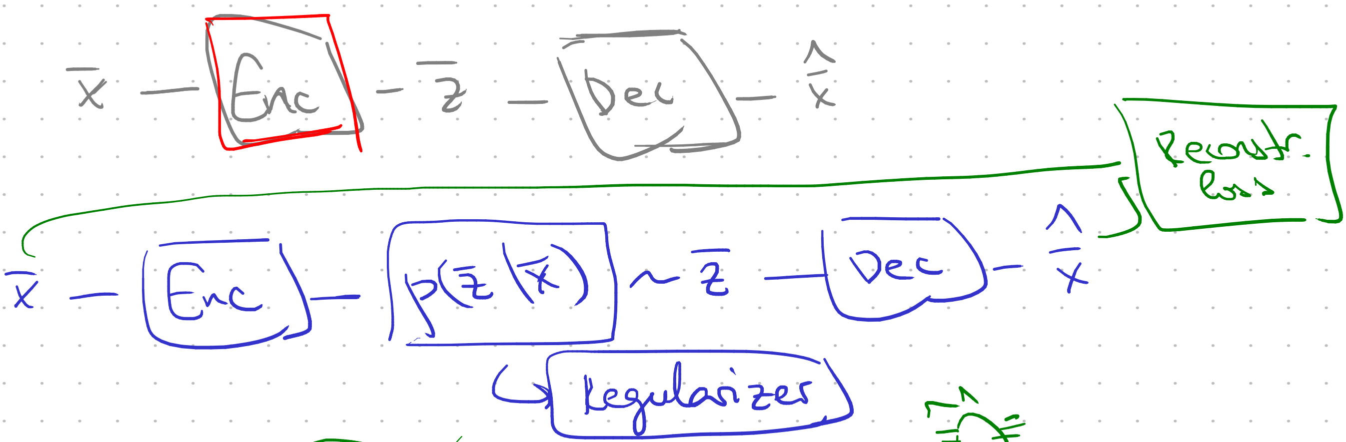
perception loss : $\|VGG(\bar{x}) - VGG(\hat{\bar{x}})\|_2^2$

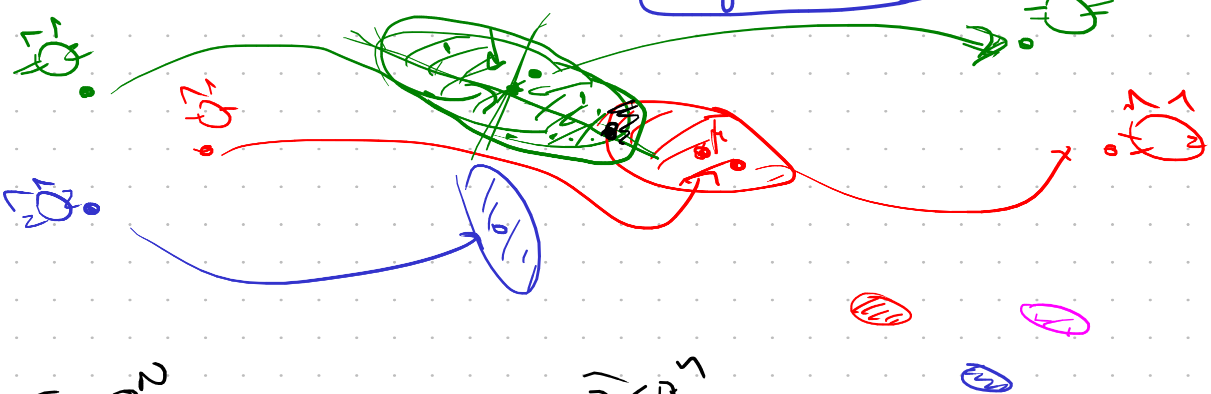


AAE :



② Variational autoencoder (VAE)



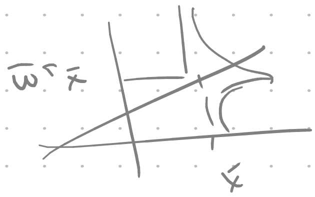


$\bar{x} \in \mathbb{R}^N$ $\bar{z} \in \mathbb{R}^n$

$$\bar{x} - \boxed{\mathrm{Enc}} - \begin{matrix}\bar{\mu}_x \\ \bar{\sigma}_x\end{matrix} \longrightarrow \bar{z} \sim \mathcal{N}(\bar{z} \mid \bar{\mu}_x, \mathrm{diag}(\bar{\sigma}_x)) - \boxed{\mathrm{Dec}} - \hat{\bar{x}}$$

$$\mathcal{N}(\bar{z} \mid \bar{\mu}_x, \bar{\sigma}_x) \approx \underbrace{\mathcal{N}(\bar{z} \mid \bar{0}, I)}_{p(\bar{z})}$$

③ VAE, take 2



$$\overbrace{\underbrace{p(\bar{z})}\ \underbrace{p(\bar{x} \mid \bar{z})}_{\text{decoder}}}^{\text{generation}} = p(\bar{x}, \bar{z}) = \underbrace{p(\bar{x})}\ \underbrace{p(\bar{z} \mid \bar{x})}_{\text{"encoder"}}$$

$$\mathcal{N}(\bar{z} \mid \bar{0}, I) \cdot \mathcal{N}(\bar{x} \mid \mathrm{Dec}(\bar{z}), c \cdot I) =$$  $$\cdot\ p(\bar{z} \mid \bar{x})$$

$$\approx q(\bar{z}) = \mathcal{N}(\bar{z} \mid \bar{\mu}_x, \bar{\sigma}_x)$$

Variational approximation

$$\log p(\bar{x}, \bar{z}) = \log p(\bar{x})\, p(\bar{z} \mid \bar{x})$$

$$\log p(\bar{x}) = \log p(\bar{x}, \bar{z}) - \log p(\bar{z} \mid \bar{x}) \quad \Big|\ \mathbb{E}_{q(\bar{z})}$$

$$\log p(\bar{x}) = \mathbb{E}_q[\log p(\bar{x}, \bar{z}) - \log p(\bar{z} \mid \bar{x})] \pm \mathbb{E}_q[\log q(\bar{z})]$$

$$\log p(\bar{x}) = \underbrace{\mathbb{E}_{q(\bar{z})}\left[\log \frac{p(\bar{x}, \bar{z})}{q(\bar{z})}\right]} + \underbrace{\mathbb{E}_{q(\bar{z})}\left[\log \frac{q(\bar{z})}{p(\bar{z} \mid \bar{x})}\right]}$$

$$\text{Const} = \underbrace{L(q)}_{\substack{\text{variational} \\ \text{lower bound}}} + \underbrace{KL(q(\bar{z}) \,\|\, p(\bar{z} \mid \bar{x}))}$$

$q \swarrow$ max

$q \to$ min

$$L(q) \xrightarrow{} \max$$

$$L(q) = \int q(\bar{z}) \log \frac{p(\bar{x}, \bar{z})}{q(\bar{z})}\, d\bar{z} = \quad \left(p(\bar{x}, \bar{z}) = p(\bar{z})\, p(\bar{x} \mid \bar{z}) \right)$$

$$= \int q(\bar z)\log p(\bar x|\bar z)\,d\bar z - \int q(\bar z)\log\frac{q(\bar z)}{p(\bar z)}\,d\bar z$$

$$= \underbrace{\mathbb{E}_{q(\bar z)}\left[\log p(\bar x|\bar z)\right]} - \underbrace{KL\left(q(\bar z)\,\|\,p(\bar z)\right)}_{L_{reg}}$$

$$\mathbb{E}_{q(\bar z)}\left[\log \mathcal{N}(\bar x\,|\,Dec(\bar z),\ c\cdot I)\right]$$

$$= \mathbb{E}_{q(\bar z)}\left[Const - \frac{1}{2c^d}\cdot\|\bar x - Dec(\bar z)\|^2\right] \approx$$

$$\approx Const - Const\cdot\sum_n \|\bar x_n - Dec(Enc(\bar x_n))\|^2 = -L_{rec}$$

$$L(q) \approx Const - \Big(Const\cdot \underbrace{\sum_n \|\bar x_n - Dec(Enc(\bar x_n))\|^2}_{(\sigma_{1\bar x}\ \cdot\ c\ \cdot\ \sigma_{d\bar x})\cdot L_{rec}} + \underbrace{KL(q\,\|\,p(\bar z))}_{L_{reg}}\Big)$$

$$q(\bar z) = \mathcal{N}(\bar z\,|\,\bar\mu_x, \underline{\underline{\Sigma_x}}), \qquad p(\bar z) = \mathcal{N}(\bar z\,|\,\bar 0, I)$$

$$KL(q\|p) = \sum_{i=1}^{d} KL(q_i\|p_i)$$

$$KL\left(\mathcal{N}(z_i|\mu_{x_i},\sigma_{x_i})\,\|\,\mathcal{N}(z_i|0,1)\right) = \mathbb{E}_{\mathcal{N}_{\bar x}}\left[\log\frac{\mathcal{N}_{\bar x}}{\mathcal{N}}\right] =$$

$$= \mathbb{E}_{\mathcal{N}_x(\bar z)}\left[-\frac{1}{2}\log\sigma_{x_i}^2 - \frac{1}{2\sigma_{x_i}^2}(z_i-\mu_{x_i})^2 + \frac{1}{2}z_i^2\right] =$$

$$= -\frac{1}{2}\log\sigma_{x_i}^2 - \frac{\mu_{x_i}^2}{2\sigma_{x_i}^2} + \mathbb{E}_{\mathcal{N}_x(\bar z)}\left[\frac{1}{2}z_i^2 + \frac{z_i\mu_{x_i}}{\sigma_{x_i}^2} - \frac{1}{2\sigma_{x_i}^2}z_i^2\right]$$

$$\mathbb{E}_{\mathcal{N}(z_i|\mu_{x_i},\sigma_{x_i}^2)}[z_i] = \mu_{x_i} \qquad \Big| \qquad \mathbb{E}_{\mathcal{N}(z_i|\mu_{x_i},\sigma_{x_i}^2)}[z_i^2] = \mu_{x_i}^2 + \sigma_{x_i}^2$$

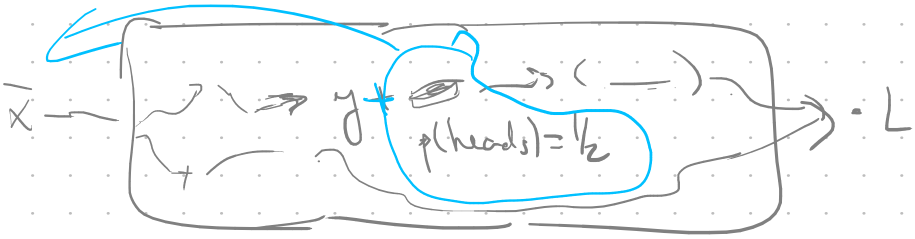
$$= -\frac{1}{2}\log\sigma_{x_i}^2 - \cancel{\frac{\mu_{x_i}^2}{2\sigma_{x_i}^2}} + \frac{1}{2}\cdot\left(\mu_{x_i}^2+\sigma_{x_i}^2\right)\left(1-\frac{1}{\sigma_{x_i}^2}\right) + \cancel{\frac{\mu_{x_i}^2}{2\sigma_{x_i}^2}} =$$

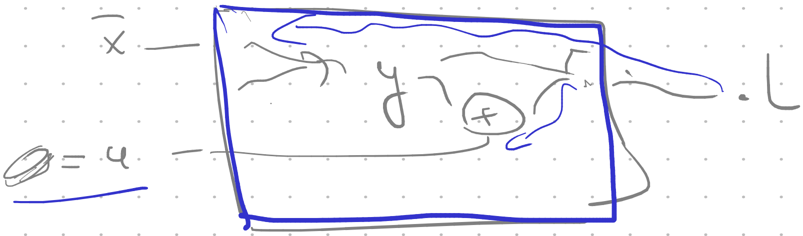
$$= -\frac{1}{2}\log\sigma_{x_i}^2 + \frac{\mu_{x_i}^2}{2} + \frac{1}{2}\sigma_{x_i}^2 - \frac{1}{2}$$

$$L_{reg} = \sum_{i=1}^{d}\left(\mu_{x_i}^2 + \sigma_{x_i}^2 - \log\sigma_{x_i}^2\right)$$

(4) Reparametrization trick

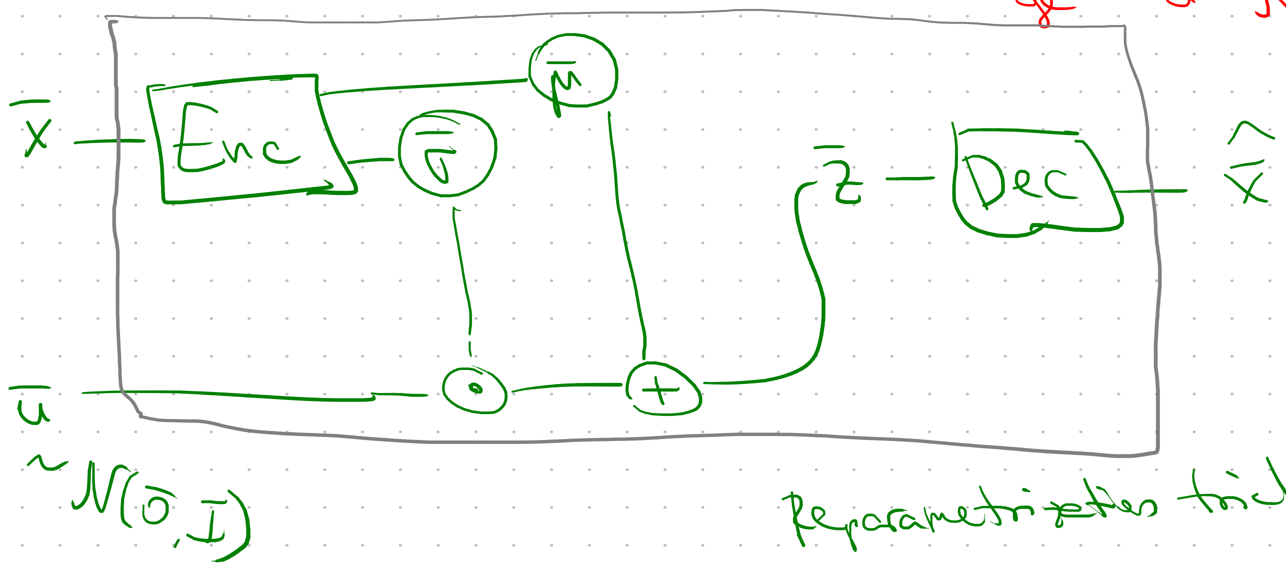
$\bar{x}$ — $\boxed{\text{Enc}}$ — $\bar{\mu}$, $\bar{\sigma}$ > $\boxed{\text{Sample}}$ — $\bar{z}$ — $\boxed{\text{Dec}}$ — $\hat{\bar{x}}$





$$\bar{z} \sim N(\bar{z} \mid \bar{\mu}, \bar{\sigma}) \iff \bar{z} = \bar{u}\odot\bar{\sigma} + \bar{\mu},$$

$$\bar{u} \sim N(\bar{u} \mid \bar{0}, I)$$



Reparametrization trick