

Hard satisfiable instances for DPLL-type algorithms

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1 Introduction

Satisfiability is one of the most popular NP-complete problems. There are two main types of algorithms for solving SAT, namely local search (for references see, for example, [3]) and DPLL-type (this type was first described in the work [5] of Davis and Putnam and [4] of Davis, Logemann and Loveland). A lot of effort has been invested in proving "less-than- 2^N " upper bounds for such algorithms. In this paper we concentrate on proving exponential lower bounds and consider two DPLL-type algorithms: GUC (Generalized Unit Clause heuristic; introduced in [2]) and Randomized GUC.

DPLL-type algorithms were historically the first "less-than- 2^N " algorithms for SAT. They receive as input a formula F in CNF with variables $x_1, \dots, x_N$. After that, a DPLL-type algorithm simplifies the input according to a certain set of *transformation rules*. If the answer now is obvious (the simplified formula is either empty or contains a pair of contradicting unit clauses), the algorithm returns an answer. In the opposite case, it chooses a literal l in the formula according to a certain heuristic. Then it constructs two formulas, one corresponding to $l := \text{true}$ and the other to $l := \text{false}$, and recursively calls itself for these two formulas (note that since we deal with the running time of the algorithm, the order in which it calls itself for these two formulas does matter). If any of the calls returns the answer "Satisfiable", the algorithm also returns this answer. Otherwise, it returns "Unsatisfiable". Therefore, such algorithms differ from each other by two procedures: one for simplifying a formula, and the other for choosing the next literal.

Superpolynomial lower bounds for regular resolution (and hence DPLL-type algorithms) are known since [10]. In [1], a probabilistic distribution of exponentially hard formulas was considered, and an exponential lower

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bound for two DPLL-type algorithms (GUC and UC) was proved. Supposedly, among them satisfiable instances should exist, but this question remains open. Exponentially hard satisfiable instances were found for local search algorithms (see [8]). However, no results have been published about exponentially hard *provably satisfiable* formulas for DPLL-type algorithms. In this paper we present such instances. Section 2 is devoted to basic definitions, in section 3 examples of hard unsatisfiable formulas are given, sections 4 and 5 contain proofs of lower bounds for the algorithms, and open questions are formulated in Section 6.

2 Preliminaries

We denote by X a set of boolean variables. The negation of a variable x is denoted by $\overline{x}$. If $U \subseteq X$, then $\overline{U} = \{\overline{x} \mid x \in U\}$. *Literals* are members of the set $X \cup \overline{X}$. A *clause* is a set of literals that does not contain simultaneously any variable together with its negation. A *formula in CNF* is a finite set of clauses. A clause is called *unit* if it consists of one literal. A literal is called *pure* with respect to a formula if the formula contains only the literal, but does not contain its negation. We denote by $PL(F)$ the collection of all pure literals in F .

An *assignment* is a finite subset $I \subseteq X \cup \overline{X}$ that does not contain any variable together with its negation. We denote by $F[I]$ a formula that results from F and an assignment $I = \{x_1, x_2, \dots, x_n\}$ after removing all clauses containing the literals x_i and deleting all occurrences of the literals $\overline{x_i}$ from the other clauses. An assignment I is said to *satisfy* the formula F , if $F[I]$ is the empty formula (that is, $F[I]$ contains no clauses).

For a formula $F(x_1, \dots, x_n)$ we construct its binary *assignment tree*. Its nodes are partial assignments for F consisting of literals $x_1, \dots, x_n$ or their negations, and the sons of a node $I = \{l_1, \dots, l_i\}$, where $l_j \in \{x_j, \overline{x_j}\}$, are the assignments $I_1 = \{l_1, \dots, l_i, x_{i+1}\}$ and $I_2 = \{l_1, \dots, l_i, \overline{x_{i+1}}\}$. Following [2], we denote by C_i^F the collection of clauses in F containing exactly i literals (we will omit the upper index if it is clear from context).

3 Hard unsatisfiable formulas

First examples of unsatisfiable formulas requiring superpolynomial time for regular resolution (shown in [6] to be equivalent to the Davis-Putnam procedure for complexity issues) appeared in [10]. Examples in that article were obtained by using boolean formulas based on graphs. Tseitin used rather simple graphs, and his bounds were improved by Galil in [6]. In [11], using

Making a choice with the GUC algorithm**Input:** A formula F in CNF.**Method:**

1. $m := \min\{i : C_i^F \neq \emptyset\}$.
2. If $m = 1$ then choose C randomly from C_1^F and set l to the only literal in C .
3. Else if $PL(F) \neq \emptyset$, choose l randomly from $PL(F)$.
4. Else choose C randomly from C_m^F , then choose l randomly from C .
5. Output l .

Figure 1: One step of the improved GUC algorithm

the graph theory results on expanders, the bounds were improved to the form of 2^{cN} , where N is the number of variables in the formula (in the future c will denote that very constant). Note that the bounds proven in the present article depend on the best known bound for unsatisfiable formulas and, therefore, will automatically improve if the above-mentioned constant c is increased.

Let us quote the following theorem from [11] (here S_m is an always existing, previously constructed in the same article formula):

Theorem 1 ([11], 5.7). *There is a constant $c > 1$ such that for sufficiently large m , any resolution refutation of S_m contains c^n distinct clauses, where S_m is of length $O(n)$, $n = m^2$.*

In [9], using a generalization of Tseitin's tautologies, the following result was established: for every $k \geq 3$, there exists a constant $c_k > 0$, $c_k = O(1/k^{1/8})$, such that every DPLL-algorithm for k -SAT has worst-case time complexity at least $\Omega(2^{N(1-c_k)})$, where N is the number of variables in the formula.

It is also worth mentioning that formulas in [9] and [11] have linear number of clauses, that is, there is a constant b such that these formulas have less than bN clauses, where N is the number of variables in them.

We denote by $G_k(y_1, \dots, y_N)$ the hard formula in k -CNF appearing in [9] with N variables $y_1, \dots, y_N$.

4 Hard formulas for GUC

The GUC algorithm is described in [2] and its procedure for making a choice is shown here on Fig.1. Essentially, it selects a random literal satisfying a clause of the smallest size. Compared to the algorithm in [2], we have added the pure literals rule to its choice heuristic, that is, if the negation of a literal does not occur in the formula, we automatically satisfy this literal. Obviously, checking for pure literals can be done in polynomial number of steps (with respect to the number of variables). It is also obvious that applying the pure literals rule cannot make the current partial assignment contradictory. Note that our bounds also hold for the original GUC algorithm (and, later, the original Randomized GUC algorithms instead of the modified version); in fact, the pure literals rule will sometime make our bounds worse.

In this article we use the backtracking implementation described, for example, in [1]. Basically, every time an algorithm splits on some variable, it makes a choice, and the number of such choices in that case measures its efficiency. When first reaching a node, algorithm marks a choice it has to make as *forced*, if it was made by using the transformation rules. In our case, such choices occur when there is either a unit clause or a pure literal in the formula. In the opposite case we will call a choice *free*. The backtracking implementation of GUC will “go down the assignment tree” until it finds a contradiction, and then backtrack to the last free choice. Then it flips the value assigned during this last free choice, marks this choice as forced, and continues. We measure the complexity of our algorithm as the number of choices (both free and forced) it makes until it finds a satisfying assignment.

Let us now proceed to proving the exponential lower bound on satisfiable formulas. Consider the following formula (we denote by $x \vee E$ the set of clauses obtained by adding x to all clauses in E):

$$F = (x_1 \vee G_k(x_{M+1}, \dots, x_{\lceil \frac{1+c_k}{c_k} M \rceil})) \wedge \\ \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_3 \vee \overline{x_4}) \wedge \dots \wedge (\overline{x_1} \vee x_{M-1} \vee \overline{x_M}) \wedge (\overline{x_1} \vee x_M \vee \overline{x_2})$$

Note that the second line corresponds to $\overline{x_1} \vee H$, where H is a formula forcing the variables $x_2, \dots, x_M$ to have equal values. Also note that while G_k is a formula in k -CNF, F is a formula in $(k+1)$ -CNF (and its first part is in $(k+1)$ -CNF). At the first step, GUC satisfies a random literal from a random clause of minimal size. With probability $\frac{1}{3}$ this literal is $\overline{x_1}$. In this case, our formula becomes $G_k(x_{M+1}, \dots, x_{\lceil \frac{1+c_k}{c_k} M \rceil})$, and the algorithm will have to make at least $\text{poly}(M)2^M$ choices to eliminate all leaves of the assignment tree ([11, Theorem 5.7]).

With probability $\frac{2}{3}$, GUC chooses another literal l to satisfy. Let $l = x_2$ (it does not matter which one we choose due to symmetry).

$$F[x_2] = (x_1 \vee G(x_{M+1}, \dots, x_{\lceil \frac{1+c_k}{c_k} M \rceil})) \wedge \\ \wedge (\overline{x_1} \vee x_3 \vee \overline{x_4}) \wedge (\overline{x_1} \vee x_4 \vee \overline{x_5}) \wedge \dots \wedge (\overline{x_1} \vee x_{M-1} \vee \overline{x_M}) \wedge (\overline{x_1} \vee x_M)$$

The formula now has a 2-clause, and during the next step GUC will either, with probability $\frac{1}{2}$, satisfy $\overline{x_1}$, thus creating a hard unsatisfiable instance, or satisfy x_M , and we are left with

$$F[x_2, x_M] = (x_1 \vee G(x_{M+1}, \dots, x_{\lceil \frac{1+c_k}{c_k} M \rceil})) \wedge \\ \wedge (\overline{x_1} \vee x_3 \vee \overline{x_4}) \wedge (\overline{x_1} \vee x_4 \vee \overline{x_5}) \wedge \dots \wedge (\overline{x_1} \vee x_{M-1})$$

Only when there are no 3-clauses left, the last remaining literal becomes a pure literal, and the last 2-clause is decided automatically. It follows by easy induction that the probability of setting $x_1 = false$ (and forcing GUC to work for the time $poly(M)2^M$) is

$$P(x_1 = false) = 1 - \frac{2}{3}2^{-M+3},$$

which tends to 1 exponentially fast as M tends to ∞ .

If we now denote by N the total number of variables in the formula, all of the above proves the following

Theorem 2. *For every $k \geq 4$ there exists a set of satisfiable formulas F_N^k in k -CNF such that the modified GUC algorithm requires to make at least $poly(N)2^{\frac{c_k-1}{1+c_k-1}N}$ choices to find a satisfying assignment, and F_N^k contains N variables and no more than aN clauses, where a is a constant not depending on N and $c_k = O(1/k^{1/8})$.*

5 Hard formulas for Randomized GUC

It might seem that we succeeded with the GUC algorithm only because of its highly determined behavior. The problem might be in the necessary *satisfying* a shortest clause. Our formula in the preceding section “tricks” GUC into the wrong subtree precisely because of this particular behavior. In this section, we present a hard satisfiable instance for a modification of the GUC algorithm, namely Randomized GUC algorithm. One step of this algorithm is shown on Fig.2. It chooses a literal randomly from the shortest

Making a choice with the Randomized GUC algorithm**Input:** A formula F in CNF.**Method:**

1. $m := \min\{i : C_i^F \neq \emptyset\}$.
2. If $m = 1$ then choose C randomly from C_1^F and set l to the only literal in C .
3. Else if $PL(F) \neq \emptyset$, choose l randomly from $PL(F)$.
4. Else choose C randomly from C_m^F , then choose l randomly from $C \cup \overline{C}$.
5. Output l .

Figure 2: One step of the Randomized GUC algorithm

clause, but also randomly chooses whether to satisfy it. For example, if the shortest clause is $a \vee b$, Randomized GUC could choose any literal of the set $\{a, b, \overline{a}, \overline{b}\}$.

Randomized GUC would break the example in the preceding section. Indeed, on the very first step it will have a chance of $\frac{1}{6}$ to set $x_1 = \text{true}$, thus reducing the formula to a very simple one. Therefore, by restarting Randomized GUC we can achieve arbitrarily high probability of success.

Let us consider the following formula (denoting $G := G_k(x_{M+1}, \dots, x_{\lceil \frac{3c_k+2}{3c_k} M \rceil})$ and assuming $3 \mid M$ without loss of generality):

$$\begin{aligned}
F = & (x_1 \vee G) \wedge (x_2 \vee G) \wedge \dots \wedge (x_M \vee G) \wedge \\
& \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_1}) \wedge (x_3 \vee \overline{x_1} \vee \overline{x_2}) \wedge \\
& \wedge (x_4 \vee \overline{x_5} \vee \overline{x_6}) \wedge (x_5 \vee \overline{x_6} \vee \overline{x_4}) \wedge (x_6 \vee \overline{x_4} \vee \overline{x_5}) \wedge \dots \\
& \dots \wedge (x_{M-2} \vee \overline{x_{M-1}} \vee \overline{x_M}) \wedge (x_{M-1} \vee \overline{x_M} \vee \overline{x_{M-2}}) \wedge (x_{M-1} \vee \overline{x_M} \vee \overline{x_{M-2}}).
\end{aligned}$$

As in the case described above, an assignment satisfies F if and only if it sets the variables $x_1, x_2, \dots, x_M$ to *true*.

The Randomized GUC algorithm will first choose a random clause among the shortest ones, that is, among the second part of our formula, and then a random literal from the chosen clause. Since all literals appear symmetrically, in fact it chooses a random literal among $x_1, \dots, x_M, \overline{x_1}, \dots, \overline{x_M}$ to satisfy with equal probabilities. First note that if it chooses any of the negative literals, $F[\overline{x_i}]$ would contain G as an independent subformula. It would take exponentially long for Randomized GUC to prove its unsatisfiability, since a contradiction can be reached only in this subformula (it is easy to see

that the rest of the clusters of three clauses cannot be reduced to an empty clause). So, with probability $\frac{1}{2}$, the desired result is achieved. Suppose it chooses x_1 (without loss of generality, because the formula is symmetrical with respect to the first M variables). The formula now contains two 2-clauses, $(\overline{x_2} \vee x_3)$ and $(\overline{x_3} \vee x_2)$. The algorithm now has to make a free choice with probability $\frac{1}{2}$ of success (that is, choosing x_2 or x_3 rather than a negation). If it succeeds, it gets a unit clause on the next step and chooses a value for the remaining variable correctly.

In short, every cluster of three 3-clauses with similar variables has a probability of $\frac{1}{4}$ of setting the correct values for its variables, and the algorithm considers these clusters one at a time, one after another. Therefore, the overall probability of success is

$$P(\forall i : 1 \leq i \leq M \ x_i = \text{true}) = \frac{1}{2} 2^{-\frac{2}{3}M}.$$

And in case of failure, the time Randomized GUC will require to prove the unsatisfiability of G is $\text{poly}(M) 2^{\frac{2}{3}M}$. All of the above proves the following

Theorem 3. *For every $k \geq 4$ there exists a set of satisfiable formulas F_N^k in k -CNF such that the Randomized GUC algorithm requires to make at least $\text{poly}(N) 2^{\frac{2c_k-1}{2+3c_k-1}N}$ choices to find a satisfying assignment, and F_N^k contains N variables and no more than aN^2 clauses, where a is a constant not depending on N and $c_k = O(1/k^{1/8})$.*

6 Further work

In this paper we proved an exponential lower bound for satisfiable formulas for two DPLL-type algorithms. However, “hard” formulas for the Randomized GUC algorithm turned out to have quadratic relationship between the number of clauses and the number of variables. It would be interesting to construct similar linear-sized formulas.

Also, apart from the unit clause and pure literal principles, a number of other heuristics is used in modern DPLL-type SAT solvers. Such heuristics include the resolution rule, “black-and-white literals” principle etc. (for more information see [3]). Similar bounds are still to be proven for algorithms employing these heuristics.

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