A Planar Lower Bound 1.1397... > 9/8 for Cost-Preserving Single-Source Unsplittable Flows

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July 29, 2026

Abstract

For a single-source unsplittable flow, consider rounding a feasible fractional flow $\mathbf{x}$ to one path per terminal while requiring both the cost constraint $\mathbf{c}^\mathsf{T}\mathbf{y} \le \mathbf{c}^\mathsf{T}\mathbf{x}$ and the overload constraint $y_a \le x_a + Cd_{\max}$ on every arc. Dinitz, Garg and Goemans proved that the congestion bound alone is achievable with $C = 1$; Goemans conjectured that cost preservation comes for free, and this was disproved in July 2026 by a seven-vertex instance with critical constant 16/15.

We give a four-terminal planar acyclic family showing that every universal constant C in the overload constraint must satisfy

$$C \;\ge\; C^\star := \frac{299 - 41\sqrt{41}}{32} \;=\; 1.139747070789\ldots \;>\; \frac{9}{8}.$$

The construction is a directed spine with two private exits per terminal and exactly two simple source-terminal paths for each terminal. An integer one-parameter subfamily has exact critical constant $182359/160000 - 1/n$, which is larger than $\frac{9}{8}$ for $n \ge 68$. We also prove that 9/8 is the exact supremum of the three-terminal template underlying the known counterexample, so a different conflict system was needed; indeed, our new counterexample places four crossing intervals on a spine and drives the conflicts through the unavoidable baseline loads of the expensive routes. The graph is small and the results are easy to verify by hand, but we additionally certify all our claims in exact rational or symbolic arithmetic by programs that rediscover all simple source-terminal paths from the raw arc list; we make the certificate programs available at `https://github.com/snikolenko/unsplittable-flows`. We conjecture that the exact supremum of forced constants is 2, and record some structural questions that stand between $C^\star$ and 2.

1 Introduction

An instance of *single-source unsplittable flow* (SSUF) consists of a digraph $D = (V, A)$, a source s, terminals $t_1, \ldots, t_k$ with demands $d_i > 0$, and nonnegative arc costs $\mathbf{c} \in \mathbb{R}^A_{\ge 0}$. Let $\mathbf{x}$ be a feasible fractional multicommodity flow sending d_i units from s to t_i, and put $d_{\max} = \max_i d_i$. An *unsplittable* flow chooses one $s \to t_i$ path for each terminal and sends all d_i units along it; we write $\mathbf{y}$ for its arc-load vector.

Dinitz, Garg, and Goemans [2] proved that some unsplittable $\mathbf{y}$ always satisfies $y_a \le x_a + d_{\max}$ on every arc. Goemans conjectured the cost-preserving strengthening: the same additive bound *together with* $\mathbf{c}^\mathsf{T}\mathbf{y} \le \mathbf{c}^\mathsf{T}\mathbf{x}$. This conjecture stood for a quarter of a century. It is known to hold when the demands form a divisibility chain (Skutella [13]; see also Morell and Skutella [9]), and for series-parallel digraphs, where Almoghrabi, Skutella, and Warode [6] proved even the two-sided form for general multiflows. Traub, Vargas Koch, and Zenklusen [17, 18] proved a cost-preserving planar bound with additive error $2d_{\max}$, and Swamy, Traub, Vargas Koch, and Zenklusen [15] gave a general transformation deriving cost-preserving guarantees from cost-free error-bounded ones; in particular, the two-sided conjecture of Morell and Skutella [9] would imply a universal cost-preserving additive constant $2d_{\max}$.

In July 2026 the exact conjecture was disproved by an explicit seven-vertex instance whose critical constant is 16/15 [11], a counterexample found in an impressively brief discussion with a frontier LLM (GPT-5.6). This result leads to the quantitative question addressed here: *how large must a universal cost-preserving additive constant be, if one exists at all?* Note that whether a finite universal constant exists for general graphs is still an open question; for planar instances 2 is known to suffice [17].

In this work, we make the following contributions.

1. An improved lower bound. Section 4 constructs a four-terminal planar acyclic family whose exact per-instance constants converge to $(299 - 41\sqrt{41})/32 > 9/8$ (Theorem 4.7), together with an integer subfamily of exact constant $182359/160000 - 1/n$, which exceeds 9/8 for $n \geq 68$ (Corollary 4.8). The underlying digraph is a directed spine with two private exits per terminal; consequently every terminal has exactly two simple source-terminal paths, and the instance is immune, by construction, to the hybrid-path ("splicing") failure mode that invalidates some naive gadget constructions. We also prove (Proposition 5.2) that the symmetric three-terminal template underlying the seven-vertex counterexample has exact supremum 9/8; the four-interval system is therefore not a direct optimization of the known example but a different conflict system.

2. Where a stronger bound would have to come from. Section 6 records what we know about the obstruction. The abstract rounding problem that we project in flow instances does not admit a universal constant at all: odd antiholes force $2 - 4/n$ and Hadamard sign systems force $\Omega(\sqrt{n})$. However, none of those systems are known to be realizable by an actual flow, and the obvious "one shared arc per set" gadgets lead to unintended source-terminal paths. We isolate this question as open Problem 6.2. We conjecture (Conjecture 6.1) that the exact supremum of forced constants is 2, the value of the planar upper bound of Traub, Vargas Koch and Zenklusen [17] and of the conditional bound of Swamy, Traub, Vargas Koch and Zenklusen [15], and that it is not attained exactly.

Figure 1 summarizes the resulting landscape.

All known counterexamples, including our new one, follow the same template. Each terminal is offered a cheap route and an expensive route; costs are calibrated so that a routing satisfies the cost constraints exactly when it uses at least a prescribed number of cheap routes; and the cheap routes are arranged so that any admissible combination of them overloads some shared arc.

Writing $z_i \in \{0, 1\}$ for the choice of terminal i and f_i for the fractional share on its second route, the overload of a shared arc e is

$$y_e - x_e = \sum_i d_i(z_i - f_i)\, \boldsymbol{\Delta}_{i,e}, \qquad \boldsymbol{\Delta}_i = \boldsymbol{\chi}(P_i^1) - \boldsymbol{\chi}(P_i^0), \tag{1}$$

where P_i^0, P_i^1 are the two routes. Everything that both routes share (a common prefix and/or a common suffix) cancels from (1). This is the key point of newly discovered counterexamples: the support of a row of Δ does not have to be an "edge" of an intended conflict graph, and a terminal

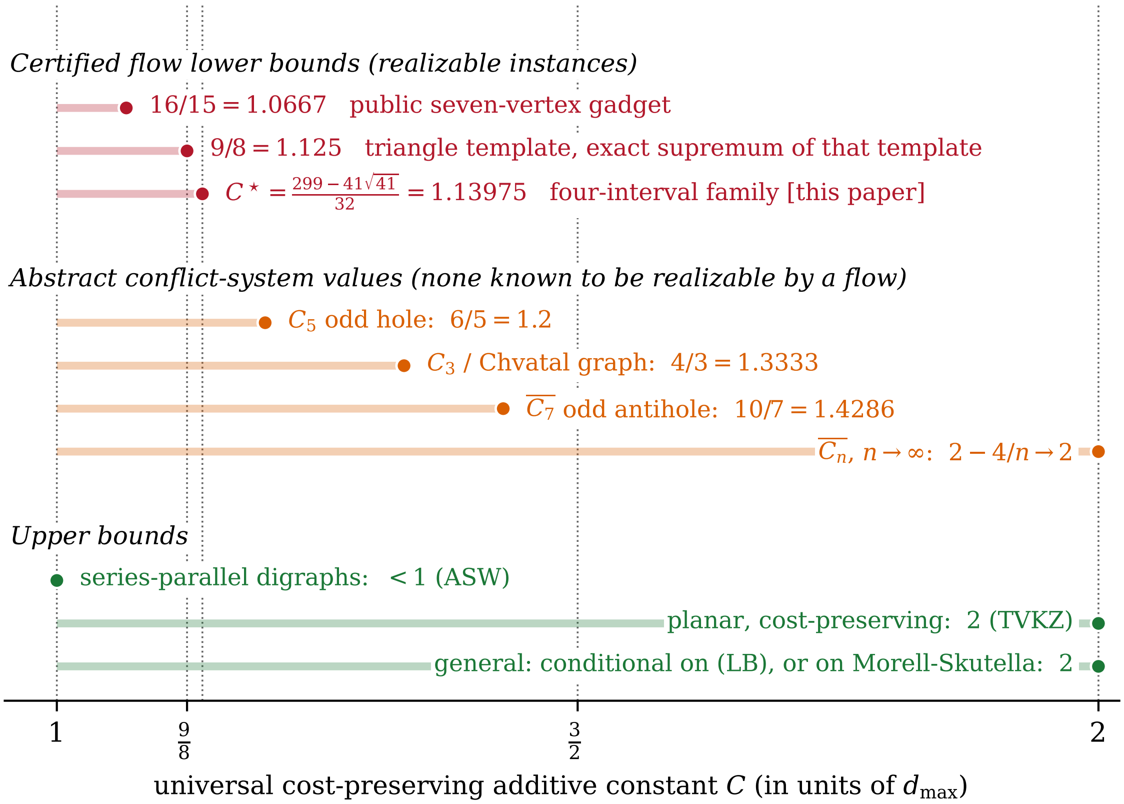

Figure 1: Bounds on the universal cost-preserving additive constant C. Top to bottom: certified lower bounds from actual flow instances; values achievable by abstract conflict systems but not (known to be) realizable by a flow instance (Problem 6.2); upper bounds. The gap between $C^\star = 1.1397\ldots$ and 2 is the main open problem.

that is not switched can still contribute to a row through the baseline load of its expensive route. The seven-vertex counterexample works exactly this way (Figure 6), and our family generalizes the mechanism to four crossing intervals on a spine.

Note that one cannot avoid this indirection. If one insists on a *faithful* realization, that is, one shared arc a_S per set S of a prescribed system, traversed by exactly the cheap routes of the members of S, then at any shared arc a prefix-suffix splice of two cheap routes produces a further cheap source-terminal path, and arc costs cannot make both splices more expensive than both originals (see Section 6). The system realized here is an interval system that contains crossing intervals $I_0 = [0, 2]$ and $I_2 = [1, 4]$, and the crossing is realized by common prefix baselines of t_2 and t_3 rather than by incidence.

AI use. This counterexample and its accompanying theoretical evaluation have been produced in close collaboration with frontier LLMs, specifically GPT-5.6 Sol, Claude Fable 5, and Claude Opus 5. While I cannot boast the same kind of "chad prompting" as Dmitry Rybin [11][1], as my discussions took longer and were more involved, I would like to express my respect for and amazement at the capabilities of frontier AI models. This is their work much more than my own.

[1]The discussion contained four short prompts, including the now-famous "You should do a breakthrough".

Verification Every numerical claim in this paper has been checked in exact rational or symbolic arithmetic by programs that rebuild each instance from its raw arc list, discover all simple source-terminal paths by depth-first search (rather than receiving the intended paths as input), and enumerate all integral routings. Section 4.5 and Appendix A describe the artifacts; we make them available at `https://github.com/snikolenko/unsplittable-flows`.

The rest of the paper is organized as follows. Section 2 briefly surveys related work, Section 3 introduces notation and preliminaries, Section 4 presents our construction, Section 5 discusses the already found counterexample and its limitations, and Section 6 discusses possible avenues for further work and directions of how one might attempt to close the gap.

2 Related work

Dinitz, Garg, and Goemans [2] introduced the additive formulation and proved the $+d_{\max}$ congestion theorem; Goemans' cost-preserving strengthening is stated there. Kolliopoulos and Stein [4] and Skutella [13] gave the classical bicriteria guarantees $y_a \le 2x_a + d_{\max}$ with $\mathbf{c}^\mathsf{T}\mathbf{y} \le 2\mathbf{c}^\mathsf{T}\mathbf{x}$ and $\mathbf{c}^\mathsf{T}\mathbf{y} \le \mathbf{c}^\mathsf{T}\mathbf{x}$ respectively; see the introduction of [12] for the exact historical statements. Note that a guarantee of the form $\mathbf{y} \le 2\mathbf{x} + d_{\max}$ yields *no* finite additive constant relative to $\mathbf{x}$; the two normalizations must not be conflated. Martens, Salazar, and Skutella [7] gave convex decompositions into unsplittable flows for rounded demands. Morell and Skutella [9] proved the one-sided lower bound $y_a \ge x_a - d_{\max}$, the two-sided $x_a/2 - d_{\max} \le y_a \le 2x_a + d_{\max}$, and the two-sided $\pm d_{\max}$ statement for divisibility chains, and conjectured the two-sided $\pm d_{\max}$ statement in general. Traub, Vargas Koch, and Zenklusen [17, 18] proved the Morell–Skutella conjecture for planar (and bounded-genus) instances, and deduced the cost-preserving planar bound $|y_a - x_a| \le 2d_{\max}$. Almoghrabi, Skutella, and Warode [6] proved that in a series-parallel digraph every multiflow is a convex combination of unsplittable multiflows with $|y_a - x_a| < d_{\max}$; averaging gives Goemans' conjecture in that class. Swamy, Traub, Vargas Koch, and Zenklusen [15] showed that any face-preserving rounding algorithm with symmetric error $\alpha d_{\max}$ yields, for every $\boldsymbol{\lambda} \in (0, 1]$, a routing with $\mathbf{c}^\mathsf{T}\mathbf{y} \le \boldsymbol{\lambda}^{-1}\mathbf{c}^\mathsf{T}\mathbf{x}$ and $|y_a - x_a| \le (1+\boldsymbol{\lambda})\alpha d_{\max}$; at $\boldsymbol{\lambda} = 1$ this converts the Morell–Skutella conjecture into a cost-preserving $2d_{\max}$ guarantee.

Recently, Rybin [11] provided a counterexample to the Dinitz-Garg-Goemans conjecture (see Section 5). The present paper is, to our knowledge, the first to give a lower bound above 9/8, and the first to isolate some structural obstructions to using set discrepancy directly.

Discrepancy. The abstract projection (1) is a one-sided weighted set-discrepancy problem with one preserved linear functional. Classical references are Beck–Fiala [1], Spencer [14] and Matoušek [8]. The three-permutation lower bound of Newman and Nikolov [10] refutes Beck's conjecture. Tijdeman's chairman assignment theorem [16] bounds the prefix discrepancy of a common-order assignment to m agents by $(1 - \frac{1}{2m-2})$; Liu and Reis [5] extended this to weighted items with the same constant, which in particular confirms the Morell–Skutella conjecture for the common-order special case ("carpooling").

3 Preliminaries

A *routing* assigns to each terminal i one simple path $s \to t_i$; its load vector is $\mathbf{y} = \sum_i d_i \boldsymbol{\chi}(P_i)$. Given a feasible fractional flow $\mathbf{x}$ and $C \ge 0$, a routing is C-*good* if $y_a \le x_a + C d_{\max}$ for all a, and *cost-good* if $\mathbf{c}^\mathsf{T}\mathbf{y} \le \mathbf{c}^\mathsf{T}\mathbf{x}$.

Definition 3.1 (per-instance constant)**.** The *critical constant* of a pair (instance, $\mathbf{x}$) is

$$C^\star(\mathbf{x}) \;:=\; \min\{\, \max_a (y_a - x_a)/d_{\max} \;:\; \mathbf{y} \text{ is a routing with } \mathbf{c}^\mathsf{T}\mathbf{y} \le \mathbf{c}^\mathsf{T}\mathbf{x}\,\}.$$

Any *universal constant* C as in Theorem 4.7 satisfies $C \ge C^\star(\mathbf{x})$ for every instance and every feasible fractional flow $\mathbf{x}$.

The following elementary identity is used constantly throughout our proofs.

Observation 3.2 (baseline cancellation). Suppose terminal i has exactly two simple paths $s \to t_i$, P_i^0 and P_i^1, the fractional flow sends $d_i f_i$ on P_i^1 and $d_i(1-f_i)$ on P_i^0, and a routing selects $P_i^{z_i}$. Let $\boldsymbol{\Delta}_i = \boldsymbol{\chi}(P_i^1) - \boldsymbol{\chi}(P_i^0) \in \{-1,0,1\}^A$. Then

$$\mathbf{y} - \mathbf{x} = \sum_i d_i(z_i - f_i)\boldsymbol{\Delta}_i.$$

In particular, every arc used by *both* routes of i (a common prefix, a common suffix, or any other shared arc) contributes nothing to $\boldsymbol{\Delta}_i$, although it does contribute to the absolute load x_a. Moreover, if $\kappa_i^r = \mathbf{c}^\mathsf{T}\boldsymbol{\chi}(P_i^r)$, then $\mathbf{c}^\mathsf{T}(\mathbf{y}-\mathbf{x}) = \sum_i d_i(\kappa_i^1 - \kappa_i^0)(z_i - f_i)$, so the cost constraint is a single signed linear functional of the vector $\big(d_i(z_i - f_i)\big)_i$.

The "exactly two paths" hypothesis is crucial to all that follows. A third simple path $s \to t_i$ would become a third integral option that cannot be encoded by the bit z_i, and all our lower bound instances have exactly two.

4 The four-interval family

4.1 The digraph

Let

$$v_0 = s \longrightarrow v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow v_4 \longrightarrow v_5$$

be a directed spine whose arcs are indexed $0,\ldots,4$ (arc j is v_jv_{j+1}). Add four terminal vertices and the eight exit arcs shown in Figure 2:

terminal	early exit	late exit
t_0	v_0t_0	v_3t_0
t_1	v_0t_1	v_5t_1
t_2	v_1t_2	v_5t_2
t_3	v_2t_3	v_4t_3

There are no other arcs.

Lemma 4.1 (exactly two paths, and the interval structure)**.** *Every terminal t_i has exactly two simple paths $s \to t_i$, which we call* early *and* late *as shown in Fig. 2. Switching terminal i from its early path to its late path adds d_i on exactly the consecutive spine interval*

$$I_0 = [0,2], \qquad I_1 = [0,4], \qquad I_2 = [1,4], \qquad I_3 = [2,3],$$

and changes only the two private exit arcs of t_i otherwise. Consequently the row support of spine arc e, $R_e = \{i : e \in I_i\}$, is

$$R_0 = \{0,1\}, \quad R_1 = \{0,1,2\}, \quad R_2 = \{0,1,2,3\}, \quad R_3 = \{1,2,3\}, \quad R_4 = \{1,2\}.$$

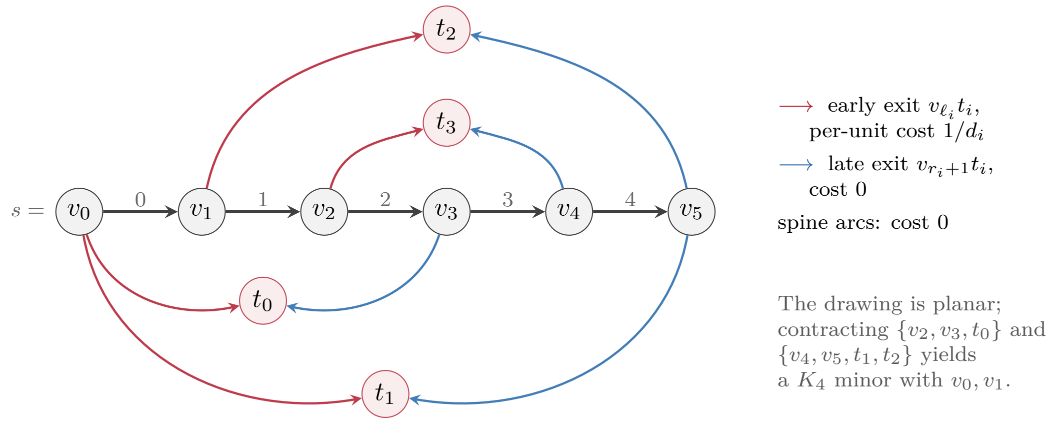

Figure 2: The four-interval digraph. Every terminal has exactly two simple paths $s \to t_i$: a spine prefix followed by its early exit, or a longer spine prefix followed by its late exit. No hybrid path exists for any choice of parameters.

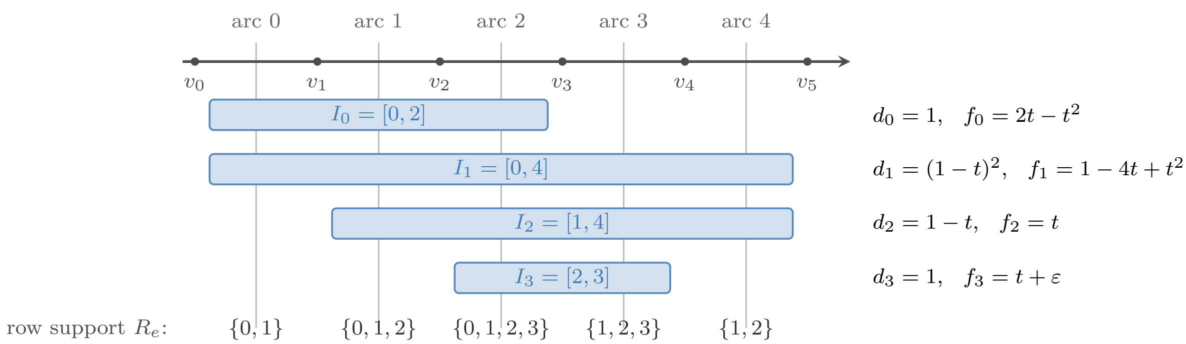

Figure 3: The interval system. Row e of the route difference matrix is supported on $R_e = \{i : e \in I_i\}$. Note that I_0 and I_2 *cross*, so the system has no coverage-faithful realization; the spine realizes it only through the common-prefix baselines of t_2 and t_3.

Proof. Obvious by construction of the graph: the only arcs leaving v_j are the spine arc j (for $j \leq 4$) and the exit arcs incident to v_j, every exit arc ends at a terminal, which has out-degree 0, so there are only two paths $s \to t_i$ for every i. The difference in the spine between them is the interval $I_i = [\ell_i, r_i]$, where ℓ_i and $r_i + 1$ are the two exit vertices. $\square$

Note that $I_0 = [0, 2]$ and $I_2 = [1, 4]$ *cross*: neither contains the other and they intersect. A system with crossing sets cannot be realized "faithfully", with one shared arc per set traversed by exactly the free routes of its members (see Problem 6.2); the spine realizes it only because the early paths of t_2 and t_3 carry *baseline* load on arcs 0 and 0, 1 (see Figure 3).

Lemma 4.2. *The underlying undirected graph is planar and has a K_4 minor; in particular it is not series-parallel.*

Proof. Planarity is exhibited by Figure 2. For the minor, consider the four vertex sets

$$\{v_0\}, \quad \{v_1\}, \quad \{v_2, v_3, t_0\}, \quad \{v_4, v_5, t_1, t_2\};$$

contracting them gives K_4. A graph is series-parallel only if it is K_4-minor-free [3]. $\square$

4.2 A two-parameter exact family

Definition 4.3 (the family $\mathcal{F}(t,\varepsilon)$)**.** Fix rational parameters

$$0 < t < 2-\sqrt{3}, \qquad 0 < \varepsilon < (1-t)^2$$

and set

$$\mathbf{d} = (1,\ (1-t)^2,\ 1-t,\ 1), \tag{2}$$

so $d_{\max} = 1$. Let the fractional flow send the following fraction of terminal i's demand on its late path:

$$\mathbf{f} = (2t-t^2,\ 1-4t+t^2,\ t,\ t+\varepsilon). \tag{3}$$

All remaining flow, $1-\mathbf{f}$, uses the early path. Put cost $1/d_i$ on terminal i's early exit arc and cost 0 on every other arc.

Lemma 4.4 (feasibility and costs)**.** *For parameters as in Definition 4.3:*

(1) $d_i \in (0,1]$ with $d_{\max} = 1$, and $\mathbf{f} \in [0,1]^4$;

(2) $\sum_{i=0}^{3} f_i = 1+\varepsilon$;

(3) every full early path has cost 1 and every late path has cost 0, so $\mathbf{c}^\mathsf{T}\mathbf{x} = \sum_i (1-f_i) = 3-\varepsilon$;

(4) an integral routing is cost-good if and only if it uses at most two early paths, equivalently at least two late paths; in particular, exactly 11 of the 16 routings are cost-good.

Proof. (1) For $0 < t < 2-\sqrt{3} < 1$ we have $0 < (1-t)^2 < 1-t < 1$, so $d_{\max} = d_0 = d_3 = 1$. For the shares, $f_0 = t(2-t) \in (0,1)$ and $f_2 = t \in (0,1)$ are clear; $f_1 = 1-4t+t^2 \geq 0$ exactly when $t \leq 2-\sqrt{3}$, and $f_1 \leq 1$ since $t^2 \leq 4t$; finally $0 < f_3 = t+\varepsilon < t+(1-t)^2 \leq 1$ because $t+(1-t)^2-1 = t^2-t = -t(1-t) < 0$.

(2) $(2t-t^2)+(1-4t+t^2)+t+(t+\varepsilon) = 1+\varepsilon$.

(3) The early path of t_i consists of zero-cost spine arcs and the single early exit arc of per-unit cost $1/d_i$; routing all d_i units along it costs 1. Late paths use only zero-cost arcs. Hence

$$\mathbf{c}^\mathsf{T}\mathbf{x} = \sum_i d_i(1-f_i)\cdot(1/d_i) = \sum_i (1-f_i) = 4-(1+\varepsilon) = 3-\varepsilon.$$

(4) An integral routing has cost equal to its number of early paths. Since $2 < 3-\varepsilon < 3$, a routing is cost-good iff it has at most two early paths. There are $\binom{4}{0}+\binom{4}{1}+\binom{4}{2} = 11$ such routings. □

By Observation 3.2 and Lemma 4.1, for a spine arc e and a routing $\mathbf{z} \in \{0,1\}^4$ (with $z_i = 1$ meaning "late"),

$$y_e - x_e = \sum_{i \in R_e} d_i(z_i - f_i), \tag{4}$$

while for an exit arc of terminal i the overload is $d_i(1-f_i)$ if $z_i = 1$ (its late exit) and $d_i f_i$ if $z_i = 0$ (its early exit), so in both cases it is at most $d_i \leq 1$.

Let

$$C(t) = (1-t)^2(1+4t-t^2) = 1+2t-8t^2+6t^3-t^4. \tag{5}$$

Lemma 4.5 (the six pair obstructions)**.** *Let $0 < t < 2-\sqrt{3}$ and $0 < \varepsilon < (1-t)^2$. The table below lists maximum overloads over all arcs for each fixed late pair, as attained on the indicated witness spine arc:*

Late pair	*Max overload*	*Witness arc*
$\{0,1\}$	$C(t)$	0
$\{0,2\}$	$C(t)$	1
$\{0,3\}$	$C(t)-\varepsilon$	2
$\{1,2\}$	$C(t)$	4
$\{1,3\}$	$C(t)-\varepsilon$	3
$\{2,3\}$	$C(t)+t(1-t)-\varepsilon$	3

Moreover, if $\varepsilon < C(t)-1$ then additionally $C(t)-\varepsilon > 1$, which holds in particular on the rectangle $\frac{7}{50} \le t \le \frac{4}{25}$, $0 < \varepsilon \le \frac{1}{68}$.

Proof. Substituting (2) and (3) into (4) gives, for each pair, a vector of five polynomials in t, ε. We verify the witness identities and then the fact that they dominate the others.

Witness identities. For the pair $\{0,1\}$ on arc 0 ($R_0 = \{0,1\}$),

$$d_0(1-f_0) + d_1(1-f_1) = (1-2t+t^2) + (1-t)^2(4t-t^2) = (1-t)^2(1+4t-t^2) = C(t),$$

the last equality by expansion. The other five identities are the same computation; for instance for $\{2,3\}$ on arc 3 ($R_3 = \{1,2,3\}$),

$$-d_1 f_1 + d_2(1-f_2) + d_3(1-f_3) = -(1-t)^2(1-4t+t^2) + (1-t)^2 + (1-t-\varepsilon),$$

which expands to $1+3t-9t^2+6t^3-t^4-\varepsilon = C(t)+t(1-t)-\varepsilon$.

Witnesses have max overload across the spine. Subtracting each competing spine coordinate from the claimed witness value yields the following table of gaps, in which every entry is a polynomial in t and ε with the stated closed form:

Late pair	arc 0	arc 1	arc 2	arc 3	arc 4
$\{0,1\}$	—	$t(1-t)$	$\varepsilon+t(2-t)$	$\varepsilon+1$	$1-t$
$\{0,2\}$	$(1-t)^2$	—	$\varepsilon+t$	$\varepsilon+t^2-t+1$	$(1-t)^2$
$\{0,3\}$	$(1-t)^2-\varepsilon$	$1-t-\varepsilon$	—	$(1-t)^2$	$(1-t)(2-t)-\varepsilon$
$\{1,2\}$	1	$t(2-t)$	$\varepsilon+t(3-t)$	$\varepsilon+t$	—
$\{1,3\}$	$1-\varepsilon$	$1+t-t^2-\varepsilon$	$t(2-t)$	—	$1-t-\varepsilon$
$\{2,3\}$	$2-t-\varepsilon$	$1+t-t^2-\varepsilon$	$t(2-t)$	—	$1-t-\varepsilon$

It is easy to see that every entry is strictly positive for $0 < t < 1$ and $0 < \varepsilon < (1-t)^2$. Indeed, entries without ε are obviously positive, as are entries of the form $\alpha+\varepsilon$ with $\alpha > 0$. As for the four entries where ε is subtracted, note that

$$(1-t)^2-\varepsilon > 0 \quad \text{and} \quad 1-t-\varepsilon \;\ge\; (1-t)-(1-t)^2 = t(1-t) > 0,$$

and then note that $1-\varepsilon$, $1+t-t^2-\varepsilon$, $2-t-\varepsilon$ and $(1-t)(2-t)-\varepsilon$ are all larger than $1-t-\varepsilon$ on $0 < t < 1$. Hence for each pair the claimed spine arc strictly dominates the other four. Figure 4 plots the six overload profiles.

Witnesses dominate private arcs. Every private exit overload is at most $\max_i d_i = 1$. Write $C(t)-1 = t\,h(t)$ with $h(t) = 2-8t+6t^2-t^3$. It is a direct computation that $h(t) > 0$ on

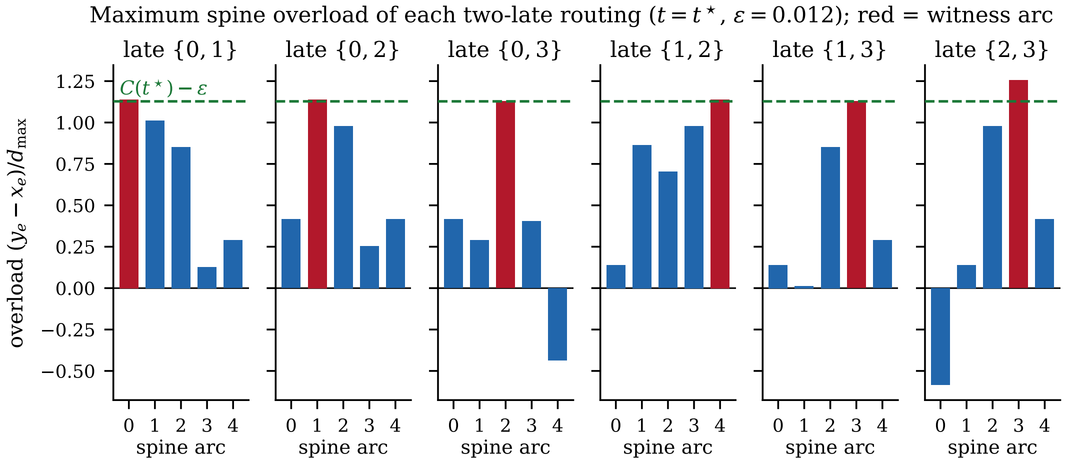

Figure 4: The overload profile of each two-late routing over the five spine arcs. The dashed line is the critical level $C(t^\star) - \varepsilon$; four of the six pairs are strictly above it, and two pairs $\{0, 3\}, \{1, 3\}$ touch it exactly.

$t \in (0, 2 - \sqrt{3}]$[2], so $C(t) > 1$ on $(0, 2 - \sqrt{3}]$. If furthermore $\varepsilon < C(t) - 1$ then $C(t) - \varepsilon > 1$, which means that no private arc can maximize overload.

In particular, on the rectangle $\frac{7}{50} \le t \le \frac{4}{25}$, $\varepsilon \le \frac{1}{68}$, direct evaluation gives $C(7/50) = 7120499/6250000 = 1.13927\ldots$ and $C(4/25) = 444969/390625 = 1.13912\ldots$; since C has a single critical point there (see the proof of Theorem 4.7) its minimum on the interval is the smaller endpoint value, so

$$C(t) - \varepsilon > 1.1391 - \tfrac{1}{68} > 1.$$

□

Lemma 4.6 (exact critical constant)**.** *Under the hypotheses of Lemma 4.5 together with $\varepsilon < C(t) - 1$, the critical constant of the instance $\mathcal{F}(t, \varepsilon)$ is exactly $C(t) - \varepsilon$.*

Proof. Lower bound. Let $\mathbf{z}$ be any cost-good routing. By Lemma 4.4(4) its late set $Z = \{i : z_i = 1\}$ has $|Z| \ge 2$. Pick a pair $\{i, j\} \subseteq Z$ and let $\mathbf{z}'$ be the routing with late set exactly $\{i, j\}$. For every spine arc e,

$$(y_e - x_e) - (\mathbf{y}'_e - \mathbf{x}'_e) = \sum_{m \in Z \setminus \{i,j\}} d_m \, [\, e \in I_m \,] \; \ge 0,$$

since all $d_m > 0$. Hence the spine-overload vector of $\mathbf{z}$ dominates that of $\mathbf{z}'$ coordinatewise, so $\max_a (y_a - x_a) \ge \max_e (\mathbf{y}'_e - \mathbf{x}'_e) \ge C(t) - \varepsilon$ by Lemma 4.5, the last inequality because $C(t) - \varepsilon$ is the smallest of the six pair values (the four others are $C(t)$ or $C(t) + t(1 - t) - \varepsilon$, both larger).

Upper bound. The routing with late set $\{0, 3\}$ is cost-good (two early paths) and, by Lemma 4.5, has maximum overload over all arcs (spine and terminal exits) of exactly $C(t) - \varepsilon$. The same holds for late set $\{1, 3\}$. □

[2]To see it formally, note that $h''(t) = 12 - 6t > 0$ on $[0, 1]$, so the derivative $h'(t) = -8 + 12t - 3t^2$ is increasing, so on $[0, 2 - \sqrt{3}]$ it is at most $h'(2 - \sqrt{3}) = -5 < 0$ and h is decreasing there; and $h(2 - \sqrt{3}) = 2 - \sqrt{3} > 0$, because $\tau = 2 - \sqrt{3}$ satisfies $\tau^2 = 4\tau - 1$, hence $\tau^3 = 4\tau^2 - \tau = 15\tau - 4$ and $h(\tau) - \tau = 2 - 9\tau + 6(4\tau - 1) - (15\tau - 4) = 0$.

4.3 Optimization and the algebraic limit

Theorem 4.7. *Suppose a universal constant C guarantees, for every instance and every feasible fractional flow $\mathbf{x}$, an unsplittable flow $\mathbf{y}$ with*

$$y_a \le x_a + C d_{\max} \quad (a \in A), \qquad \mathbf{c}^\mathsf{T}\mathbf{y} \le \mathbf{c}^\mathsf{T}\mathbf{x}.$$

Then

$$C \;\ge\; C^\star := \frac{299 - 41\sqrt{41}}{32} = 1.139747070789\ldots \;>\; \frac{9}{8}.$$

Moreover $C^\star$ is the exact supremum of the critical constants of the family $\mathcal{F}(t, \varepsilon)$, not attained by any single member.

Proof. Differentiating (5),

$$C'(t) = 2 - 16t + 18t^2 - 4t^3 = -2(t-1)(2t^2 - 7t + 1),$$

so the roots of C' are $t = 1$ and $t = \frac{7\pm\sqrt{41}}{4}$. Only

$$t^\star = \frac{7 - \sqrt{41}}{4} = 0.149218940641\ldots$$

lies in $(0, 2-\sqrt{3}) = (0, 0.267949\ldots)$, and $C''(t^\star) = -16 + 36t^\star - 12(t^\star)^2 = -10.895\ldots < 0$, so $t^\star$ is the unique maximizer of C on the feasible interval. We can reduce C modulo the minimal polynomial $2t^2 - 7t + 1$ of $t^\star$ to

$$C(t) \;\equiv\; \frac{3}{8} + \frac{41}{8}\,t \pmod{2t^2 - 7t + 1},$$

and therefore

$$C(t^\star) = \frac{3}{8} + \frac{41}{8} \cdot \frac{7 - \sqrt{41}}{4} = \frac{299 - 41\sqrt{41}}{32} = 1.139747070789\ldots > 9/8.$$

Now choose rational $t_m \to t^\star$ with $0 < t_m < 2 - \sqrt{3}$, and rational $\varepsilon_m \to 0$ with $0 < \varepsilon_m < \min\{(1-t_m)^2,\, C(t_m) - 1\}$. By Lemma 4.6 the instance $\mathcal{F}(t_m, \varepsilon_m)$ has critical constant exactly $C(t_m) - \varepsilon_m$, and $C(t_m) - \varepsilon_m \to C(t^\star)$. Multiplying all demands, path-flow amounts and costs by a common denominator makes them integral and doesn't change the normalized overload. Hence, any universal constant C should satisfy $C \ge \sup_m (C(t_m) - \varepsilon_m) = C^\star$. Note that no member attains $C^\star$: a member has critical constant $C(t) - \varepsilon < C(t^\star)$ because it needs $\varepsilon > 0$. □

Figure 5 plots the envelope.

4.4 An explicit integer subfamily

For a compact exact certificate, take $t = 3/20$ and $\varepsilon = 1/n$, and scale by $d_{\max} = 160000n$.

Corollary 4.8. *For every integer $n \ge 68$, consider the digraph of Figure 2 with demands*

$$(d_0, d_1, d_2, d_3) = (160000n,\ 115600n,\ 136000n,\ 160000n),$$

late-path amounts

$$(u_0, u_1, u_2, u_3) = (44400n,\ 48841n,\ 20400n,\ 24000n + 160000),$$

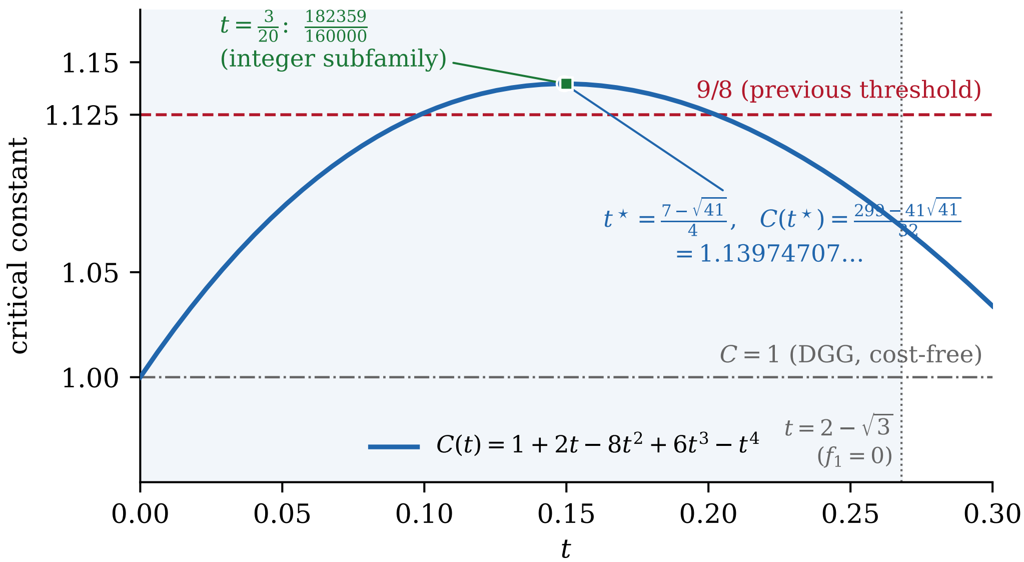

Figure 5: The envelope $C(t)$ of the four-interval family. The shaded region is the feasible parameter interval $0 < t < 2 - \sqrt{3}$, beyond which the share $f_1 = 1 - 4t + t^2$ becomes negative. The maximum is at $t^\star = (7 - \sqrt{41})/4$; the marked rational point $t = 3/20$ yields the integer subfamily of Corollary 4.8.

early-path amounts $d_i - u_i$, and per-unit costs $(289, 400, 340, 289)$ on the four early exits, zero elsewhere. Then every full early path costs $W = 46\,240\,000\,n$, $\mathbf{c}^\mathsf{T}\mathbf{x} = (3 - 1/n)W$, and the exact critical constant is

$$C_n = \frac{182359}{160000} - \frac{1}{n}.$$

It exceeds $9/8$ exactly when $1/n < 2359/160000$, i.e. for every $n \geq 68 = \lceil 160000/2359 \rceil$, and

$$C_{68} = \frac{3060103}{2720000} = \frac{9}{8} + \frac{103}{2720000}.$$

Proof. This is $\mathcal{F}(3/20, 1/n)$ scaled by $160000n$. Indeed

$$\mathbf{d} = (1, (17/20)^2, 17/20, 1) = (1, \tfrac{289}{400}, \tfrac{17}{20}, 1)$$

scales to the stated integers, and

$$\mathbf{f} = (\tfrac{111}{400}, \tfrac{169}{400}, \tfrac{3}{20}, \tfrac{3}{20} + \tfrac{1}{n})$$

gives $u_i = d_i f_i$ as stated. Each product $d_i c_i$ equals $160000n \cdot 289 = W$, so all early paths cost the same. The parameters satisfy $0 < 3/20 < 2 - \sqrt{3}$ and

$$0 < 1/n \leq 1/68 < \min\{(17/20)^2,\, C(3/20) - 1\} = \min\{0.7225,\, 0.13974\ldots\},$$

so Lemma 4.6 applies and gives critical constant $C(3/20) - 1/n$. Finally,

$$C(3/20) = 1 + \tfrac{3}{10} - 8 \cdot \tfrac{9}{400} + 6 \cdot \tfrac{27}{8000} - \tfrac{81}{160000} = \tfrac{182359}{160000}.$$

□

Appendix B lists the resulting overload table in integers.

The demand ratios $400 : 289 : 340 : 400$ contain no divisibility relation, so the family respects the chain firewall of [13, 9]; planarity keeps it consistent with the $2d_{\max}$ upper bound of [17]; and the K_4 minor (Lemma 4.2) excludes the series-parallel class of [6].

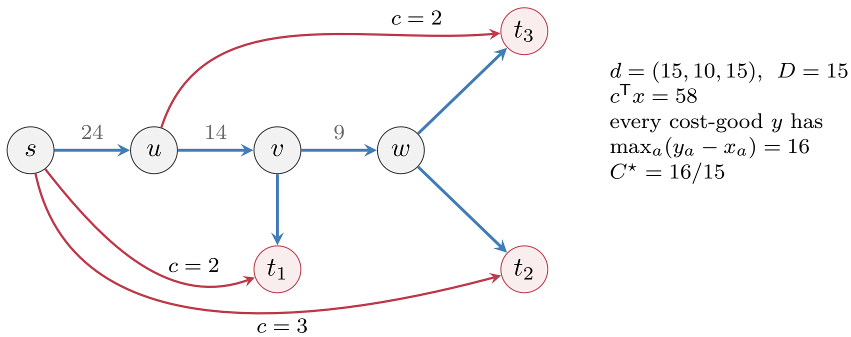

Figure 6: The seven-vertex instance with critical constant 16/15. Numbers on the spine are the fractional loads x_a. Note that the expensive route of t_3 is not a private arc: this baseline load creates the third pairwise conflict.

Remark 4.9 (Discovery). The instance was found by optimizing a floating-point circular weighted-prefix abstraction whose witness supports turned out to have the consecutive-ones property; deleting a load-irrelevant fifth terminal produced the four intervals above. The first certified value on this topology was $8867/7800 = 9/8 + 23/1950$ at the parameter point $\mathbf{d} = (1, \frac{37}{50}, \frac{7}{8}, 1)$, $\mathbf{f} = (\frac{13}{50}, \frac{6}{13}, \frac{11}{75}, \frac{2}{15})$; the two-parameter family subsumes it, and we give the best found value above.

4.5 Exact verification

Four independently written programs accompany this paper; see Appendix A. The primary verifier constructs the raw arc list and discovers every simple path $s \to t_i$ path by depth-first search; it does not receive the two intended paths as input. It finds exactly two paths per terminal, enumerates all $2^4 = 16$ integral routings, retains the eleven cost-good routings, and computes loads and costs in exact rational arithmetic, confirming Corollary 4.8 at $n = 68, 100, 1000, 10^6$ and confirming that $n = 67$ does *not* exceed 9/8 (its constant is $12058053/10720000 = 9/8 - 1947/10720000$). A second implementation represents every overload as an affine function of n and proves Corollary 4.8 for all $n \geq 68$ symbolically. A third verifies the six witness identities of Lemma 4.5 exactly, certifies each of the 36 dominance gaps on the whole parameter rectangle by a Bernstein certificate, and confirms the factorization of C' and the closed form of $C(t^\star)$. A fourth, written from the prose description alone, repeats the blind path discovery and full routing enumeration, and independently reproduces Propositions 5.1 and 5.2.

5 Previous counterexample: the seven-vertex instance and the exact value of the previous template

For calibration, we restate the known counterexample and determine the exact supremum of its general template. Figure 6 shows the instance of [11].

Proposition 5.1. *For the instance of Figure 6, namely demands* $(15, 10, 15)$*, fractional amounts* $5, 4, 5$ *on the free paths* $suvt_1$*,* $suvwt_2$*,* $suvwt_3$ *and* $10, 6, 10$ *on* st_1*,* st_2*,* sut_3*, and costs* $2, 3, 2$ *per unit on* st_1, st_2, ut_3*, there are exactly two simple paths per terminal,* $\mathbf{c}^\mathsf{T}\mathbf{x} = 58$*, exactly four routings are cost-good, and the critical constant is exactly* $16/15$*.*

Proof. Each terminal has exactly the two listed paths (immediate from the arc list; t_3 has sut_3 and $suvwt_3$). Each full expensive route costs 30, so $\mathbf{c}^\mathsf{T}\mathbf{x} = 30(\frac{2}{3} + \frac{3}{5} + \frac{2}{3}) = 58$, and a routing is cost-good iff it uses at most one expensive route, i.e. at least two free routes. The three two-free routings have excess 16 on vw, uv, su respectively, and the all-free routing has excess 24 on su; since $d_{\max} = 15$, the minimum over cost-good routings is 16/15. All comparisons are integral. $\square$

The three free paths of Figure 6 form a *prefix tree*, so the conflict system is not the triangle it appears to be: the three pairwise conflicts arise because the expensive route sut_3 loads the shared arc su. The next proposition shows this template cannot be pushed past 9/8, which is why a new system was needed.

Proposition 5.2 (exact supremum of the symmetric triangle template)**.** *Consider the template of Figure 6 with normalized outer demands $d_1 = d_3 = 1$, middle demand $d_2 = b \in [0, 1]$, free-path shares (r, σ, r), equal full expensive-route costs, and cost margin $2r + \sigma = 1 + \eta$ with $\eta > 0$. Then the critical constant of every member is at most $(3 - \eta)^2/8 < 9/8$, and the bound is attained by*

$$b = \frac{3 - \eta}{4}, \qquad r = \frac{1 + \eta}{4}, \qquad \sigma = \frac{1 + \eta}{2}.$$

Therefore, 9/8 is the exact supremum of the template (not attained in the family). The integer members $\eta = 1/n$ have exact critical constant $(3n - 1)^2/(8n^2)$.

Proof. For $0 < \eta \leq 1$ the cost-good routings are exactly those with at least two free choices. Direct subtraction on the three shared arcs gives two distinct pair obstructions,

$$\begin{aligned} A &= 2 - 2r - b\sigma \quad \text{(the outer pair, on } uv\text{)}, \\ B &= 1 + b(1 - \sigma) - r \quad \text{(an adjacent pair, on } su \text{ or } vw\text{)}, \end{aligned}$$

and every private-arc overload is at most 1. Substituting $\sigma = 1 + \eta - 2r$ gives $A - B = 1 - b - r$. For fixed r, A is nonincreasing in b (as $\sigma \geq 0$) and B is nondecreasing in b (as $1 - \sigma \geq 0$), so $\min\{A, B\}$ is maximized where they are equal, namely at $b = 1 - r$, where the common value is

$$F_\eta(r) = (1 - r)(1 - \eta + 2r).$$

The identity

$$\frac{(3 - \eta)^2}{8} - F_\eta(r) = 2\Big(r - \frac{1 + \eta}{4}\Big)^2 \;\geq\; 0$$

gives $\min\{A, B\} \leq (3 - \eta)^2/8$, with equality exactly at $r = (1 + \eta)/4$. Since $\eta > 0$, $(3 - \eta)^2/8 < 9/8$, and letting $\eta \downarrow 0$ shows the supremum is 9/8. For $\eta > 1$ only the all-free routing is cost-good and its central overload is $2 - 2r + b(1 - \sigma) \leq 2 - \eta < 1$, using $(2 - \eta) - \big(2 - 2r + b(1 - \sigma)\big) = (1 - b)(1 - \sigma) \geq 0$. For the integer members put $\eta = 1/n$ and clear denominators: $d_1 = d_3 = 8n^2$, $d_2 = 2n(3n - 1)$, free amounts $2n(n + 1), (3n - 1)(n + 1), 2n(n + 1)$, expensive per-unit costs $3n - 1, 4n, 3n - 1$; the common witness excess is $(3n - 1)^2$, and we get $(3n - 1)^2/(8n^2)$. $\square$

Proposition 5.2 also shows that parallel or serial composition cannot amplify constants: choosing a cost-nonincreasing rounding independently in each component of a disjoint union gives the *maximum*, not the sum, of the component congestions, and the same holds for a serial composition whose set of paths is the Cartesian product of local fragments.

6 Discussion, open problems, and conclusion

6.1 The conjectured value

Conjecture 6.1. *The exact supremum of forced cost-preserving additive constants for single-source unsplittable flow is* 2*, and it is not attained: every instance admits an unsplittable* $\mathbf{y}$ *with* $\mathbf{y} \leq \mathbf{x} + 2d_{\max}\mathbf{1}$ *and* $\mathbf{c}^\mathsf{T}\mathbf{y} \leq \mathbf{c}^\mathsf{T}\mathbf{x}$*, while instances exist forcing constants arbitrarily close to* 2*.*

Three independent pieces of evidence point at Conjecture 6.1. First, the planar theorem of Traub, Vargas Koch and Zenklusen [17, 18] gives $|y_a - x_a| \leq 2d_{\max}$ with cost preservation unconditionally for planar instances; note that every instance in this paper, and the seven-vertex one, is planar. Second, the transformation of Swamy, Traub, Vargas Koch and Zenklusen [15] turns the two-sided cost-free conjecture of Morell and Skutella [9] into a cost-preserving $2d_{\max}$ guarantee in general. Third, in the abstract rounding model that flow instances project to (Section 6.2 below), obstructions witnessed by *pairs* of terminals cannot exceed $2d_{\max}$ at all, because such an obstruction has value $d_i(1-f_i)+d_j(1-f_j) \leq 2d_{\max}$; and 2 is approached there by odd antiholes. Closing the gap between $C^\star = 1.1397\ldots$ and 2 is the main remaining open problem.

6.2 The abstract projection, and why it does not settle the question

By Observation 3.2, an instance in which every terminal has exactly two routes projects onto the following purely combinatorial problem. Fix a set system $\mathcal{H}$ on $[k]$, demands $0 < d_i \leq d_{\max}$, shares $\mathbf{f} \in [0,1]^k$ and weights w; find $\mathbf{z} \in \{0,1\}^k$ with

$$\sum_{i \in S} d_i(z_i - f_i) \leq C d_{\max} \quad (S \in \mathcal{H}), \qquad \sum_i w_i(z_i - f_i) \geq 0, \tag{6}$$

where the second inequality encodes cost preservation. This abstract problem is unbounded: it is essentially one-sided set discrepancy with one preserved linear functional, and there exist standard discrepancy constructions such as Hadamard sign systems or the three-permutation prefix system of Newman and Nikolov [10] that yield $C \to \infty$. Even much weaker conflict systems already reach the conjectured ceiling: odd antiholes $\overline{C_n}$ force $2 - 4/n$ and complete graphs $2 - 2/n$ in the abstract model.

So the abstract problem is not the obstruction; the problem is that it is hard to *realize* these constructions. The naive way to implement a prescribed system is to give each set S one shared arc a_S traversed by exactly the free routes of the members of S.

But it is easy to see that this does not work for a system with two crossing sets A, B. Pick $r \in A \cap B$ and $j \in B \setminus A$. The free route P_r contains both a_A and a_B; if a_A precedes a_B on it, splice the prefix of P_j up to and including a_B with the suffix of P_r after a_B. The result is a zero-cost walk $s \to t_r$ that misses a_A, and erasing cycles leaves a genuine zero-cost path $s \to t_r$ missing a_A, which is an unintended route that breaks the intended encoding. Note that fixing costs cannot repair this problem: if $P_i = A_i \circ Q \circ B_i$ and $P_j = A_j \circ Q \circ B_j$ share a subpath Q, then for any arc-additive cost vector

$$c(A_i \circ Q \circ B_j) + c(A_j \circ Q \circ B_i) = c(P_i) + c(P_j),$$

so the two splices cannot both be strictly more expensive than both originals.

The four-interval family that we have presented sidesteps this entirely: it is splice-free by construction (Lemma 4.1), and it realizes crossing intervals not by incidence but through the baseline loads of early routes.

The splicing above, however, is a simple example of a general problem with devising gadgets for this problem: gadgets designed on an abstract conflict system routinely acquire extra paths, and an extra path is an extra integral option that the intended analysis does not see. So in general, every candidate must be rebuilt from its raw arc list and re-checked against *all* simple source-terminal paths to verify that nothing unintended has slipped in. So even though the graph presented in this work is very simple, we have done it in the programmatic verification part for our counterexample.

6.3 Open problems for further work

Open Problem 6.2 (realizability)**.** Characterize the conflict systems that arise as route-difference systems of a single-source instance in which every terminal has exactly two simple paths, and determine whether any of them forces a constant above $C^{\star}$. In particular: is there such a family with certified constant above 3/2, or approaching 2?

Open Problem 6.3. Determine the supremum of exact constants over the *spine-with-exits* class: a directed path plus private exit arcs, with arbitrarily many exits per terminal. Every instance in the class is splice-free and exactly decidable, so this is a concrete finite-dimensional optimization. A ceiling theorem for the class would be the first upper bound below 2 for a nontrivial family of realizable systems, while any value above $C^{\star}$ would improve Theorem 4.7.

Open Problem 6.4. Prove *any* unconditional universal constant with exact cost preservation for general graphs. None is currently known: the planar bound of [17] and the series-parallel theorem of [6] are the only unconditional cost-preserving additive guarantees available, and the general case rests on the Morell–Skutella conjecture via [15].

References

[1] József Beck and Tibor Fiala. "integer-making" theorems. *Discrete Applied Mathematics*, 3(1):1–8, 1981.

[2] Yefim Dinitz, Naveen Garg, and Michel X. Goemans. On the single-source unsplittable flow problem. *Combinatorica*, 19(1):17–41, 1999.

[3] Richard J. Duffin. Topology of series-parallel networks. *Journal of Mathematical Analysis and Applications*, 10(2):303–318, 1965.

[4] Stavros G. Kolliopoulos and Clifford Stein. Approximation algorithms for single-source unsplittable flow. *SIAM Journal on Computing*, 31(3):919–946, 2002.

[5] Siyue Liu and Victor Reis. Weighted chairman assignment and flow-time scheduling. In *17th Innovations in Theoretical Computer Science Conference (ITCS 2026)*, volume 362 of *LIPIcs*, pages 98:1–98:18, 2026. arXiv:2511.18546.

[6] Mohammed Majthoub Almoghrabi, Martin Skutella, and Philipp Warode. Integer and unsplittable multiflows in series-parallel digraphs. In *Integer Programming and Combinatorial Optimization (IPCO 2025)*, volume 15620 of *Lecture Notes in Computer Science*, pages 427–441, 2025. arXiv:2412.05182.

[7] Maren Martens, Fernanda Salazar, and Martin Skutella. Convex combinations of single source unsplittable flows. In *Algorithms – ESA 2007*, volume 4698 of *Lecture Notes in Computer Science*, pages 395–406, 2007.

[8] Jiří Matoušek. *Geometric Discrepancy: An Illustrated Guide*, volume 18 of *Algorithms and Combinatorics*. Springer, 1999.

[9] Sarah Morell and Martin Skutella. Single source unsplittable flows with arc-wise lower and upper bounds. *Mathematical Programming*, 192(1–2):477–496, 2022. Preliminary version in IPCO 2020, LNCS 12125, 339–350.

[10] Alantha Newman and Aleksandar Nikolov. A counterexample to Beck's conjecture on the discrepancy of three permutations, 2011. arXiv:1104.2922.

[11] Dmitry Rybin. A counterexample to Goemans' cost-preserving unsplittable flow conjecture. Announcement, 22 July 2026; record and independent restatement at https://mathlab.drummerduck.com/p/goemans-unsplittable-flow and https://vibemathed.com/problem/dinitz-garg-goemans-unsplittable-flow, 2026. Seven-vertex instance with $c^\mathsf{T} x = 58$ and every congestion-good routing of cost at least 60; not peer-reviewed.

[12] Fernanda Salazar and Martin Skutella. Single-source k-splittable min-cost flows. *Operations Research Letters*, 37(2):71–74, 2009.

[13] Martin Skutella. Approximating the single source unsplittable min-cost flow problem. *Mathematical Programming*, 91(3):493–514, 2002.

[14] Joel Spencer. Six standard deviations suffice. *Transactions of the American Mathematical Society*, 289(2):679–706, 1985.

[15] Chaitanya Swamy, Vera Traub, Laura Vargas Koch, and Rico Zenklusen. Unsplittable cost flows from unweighted error-bounded variants. In *Proceedings of the 2026 Symposium on Simplicity in Algorithms (SOSA)*, pages 512–523. SIAM, 2026. arXiv:2510.21287.

[16] Robert Tijdeman. The chairman assignment problem. *Discrete Mathematics*, 32(3):323–330, 1980.

[17] Vera Traub, Laura Vargas Koch, and Rico Zenklusen. Single-source unsplittable flows in planar graphs. In *Proceedings of the 2024 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 639–668, 2024. arXiv:2308.02651.

[18] Vera Traub, Laura Vargas Koch, and Rico Zenklusen. Single-source unsplittable flows in planar and bounded-genus graphs. *Mathematical Programming*, 2026.

A Verification artifacts

The repository accompanying this paper contains:

(1) the primary exact verifier: blind depth-first path discovery from the raw arc list, full routing enumeration, `fractions.Fraction` arithmetic; it checks Corollary 4.8 at $n = 68, 100, 1000, 10^6$ and the refutation at $n = 67$, and reproduces Propositions 5.1 and 5.2;

(2) the affine-in-n symbolic auditor proving Corollary 4.8 for all $n \geq 68$;

(3) the symbolic envelope auditor for Lemma 4.5 and Theorem 4.7, including exact Bernstein certificates for all 36 dominance gaps on the parameter rectangle, the factorization of C', the location of $t^\star$, and the closed form of $C(t^\star)$;

(4) an independently written second pair of auditors reproducing (i) and (iii) from the prose description alone.

B The exact overload table of the integer subfamily

For the reader who prefers integers, here is the complete overload table of Corollary 4.8 in units of $d_{\max} = 160000n$. Recall $\mathbf{d}/d_{\max} = (1, \frac{289}{400}, \frac{17}{20}, 1)$ and $\mathbf{f} = (\frac{111}{400}, \frac{169}{400}, \frac{3}{20}, \frac{3}{20} + \frac{1}{n})$.

late pair	maximum overload	witness arc	tight?
$\{0,1\}$	$182359/160000$	v_0v_1	
$\{0,2\}$	$182359/160000$	v_1v_2	
$\{0,3\}$	$182359/160000 - 1/n$	v_2v_3	✓
$\{1,2\}$	$182359/160000$	v_4v_5	
$\{1,3\}$	$182359/160000 - 1/n$	v_3v_4	✓
$\{2,3\}$	$202759/160000 - 1/n$	v_3v_4	

Every private-exit overload is at most 1, and every routing with three or four late terminals dominates one of the six rows coordinatewise on the spine. Hence the critical constant is $182359/160000 - 1/n$, attained exactly by the late sets $\{0,3\}$ and $\{1,3\}$.